Head Starts and Doomed Losers: Contest via Search*

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Abstract

This paper studies the effects of head starts in innovation contests. We analyze a two-firm winner-takes-all contest in which each firm decides when to stop a privately observed search for innovations (with recall). The firm with a superior innovation at the outset has a head start. The firm with the most successful innovation at a common deadline wins. We find that a large head start guarantees a firm victory without incurring cost. However, a medium-sized head start ensures defeat for the firm if the deadline is sufficiently long. In the latter case, the competitor wins the entire rent of the contest. The head start firm may still increase its expected payoff by discarding its initial innovation in order to indicate a commitment to search. The effects of early stage information disclosure and cost advantages are studied, respectively.

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Introduction

> "[U]nfortunately, for every Apple out there, there are a thousand other companies . . . like Woolworth, Montgomery Ward, Borders Books, Blockbuster Video, American Motors and Pan Am Airlines, that once 'ruled the roost' of their respective industries, to only get knocked off by more innovative competitors and come crashing down." (*Forbes*, January 8, 2014)

This paper studies innovation contests, which are widely observed in a variety of industries.[1] In many innovation contests, some firms have head starts: one firm has a more advanced existing technology than its rivals at the outset of a competition. The opening excerpt addresses a prominent phenomenon that is often observed in innovation contests: companies with a head start ultimately lose a competition in the long run. It seems that having a head start sometimes results in being trapped. The failure of Nokia, the former global mobile communications giant, to compete with the rise of Apple's iPhone is one example. James Surowiecki (2013) pointed out that Nokia's focus on (improving) hardware, its existing technology, and neglect of (innovating) software contributed to the company's downfall. In his point of view, this was "a classic case of a company being enthralled (and, in a way, imprisoned) by its past success" (*New Yorker Times*, September 3, 2013).

Motivated by these observations, we investigate the effects of head starts on firms' competition strategies and payoffs in innovation contests. Previous work on innovation contests focuses on reduced form games and symmetric players, and previous work on contests with head starts considers all-pay auctions with either sequential bidding or simultaneous bidding. By contrast, we consider a stochastic contest model in which one firm has a superior existing innovation at the outset of the contest and firms' decisions are dynamic. The main contribution of our paper is the identification of the long-run effects of a head start. In particular, in a certain range of the head start value, the *head start firm* becomes the ultimate loser in the long run and its competitor (or competitors) benefits greatly from its initial apparent "disadvantage". The key insight to the above phenomenon is that a large head start (e.g., a patent) indicates a firm's demise as an innovator.

Specifically, the model we develop in Section 1 entails two firms and one fixed prize. At the beginning of the game, each firm may or may not have an initial innovation.

[1] For instance, prior to World War II, the U.S. Army Air Corps regularly sponsored prototype tournaments to award a production contract to the winning manufacturers; the Federal Communication Commission held a tournament to determine the American broadcast standard for high-definition television; IBM sponsors annual tournaments in which the winning contestants receive grants to develop their projects for commercial use; in professional sports, clubs or franchises search for competent free agents before new seasons start. See Taylor (1995), Halac et al. (2015), or the website of *InnoCentive* for more examples.

Whether a firm has an initial innovation, as well as the value of the initial innovation if this firm has one, is common knowledge. If a firm conducts a search for innovations, it incurs a search cost. As long as a firm continues searching, innovations arrive according to a Poisson process. The value of each innovation is drawn independently from a fixed distribution. The search activity and innovation process of each firm are privately observed. At any time point before a common deadline, each firm decides whether to stop its search process. At the deadline, each firm releases its most effective innovation to the public, and the one whose released innovation is deemed superior wins the prize.

First, we consider equilibrium behavior in the benchmark case, in which no firm has any innovation initially, in Section 2. We divide the deadline-cost space into three regions (as in figure 1). For a given deadline, (1) if the search cost is relatively high, there are two equilibria, in each of which one firm searches until it discovers an innovation and the other firm does not search; (2) if the search cost is in the middle range, each firm searches until it discovers an innovation; (3) if the search cost is relatively low, each firm searches until it discovers an innovation with a value above a certain positive cut-off value. In the third case, the equilibrium cut-off value strictly increases as the deadline extends and the arrival rate of innovations increases, and it strictly decreases as the search cost increases.

We then extend the benchmark case to include a head start: the head start firm is assigned a better initial innovation than its competitor, called the latecomer. Section 3 considers equilibrium behavior in the case with a head start and compares equilibrium payoffs across firms, and Section 4 analyzes the effects of a head start on each firm's equilibrium payoff.

Firms' equilibrium strategies depend on the value of the head starter's initial innovation (head start). Our main findings concern the case in which the head start lies in the middle range. In this range, the head starter loses its incentive to search because of its high initial position. The latecomer takes advantage of that and searches more actively, compared to when there is no head start.

An immediate question is: who does the head start favor? When the deadline is short, the latecomer does not have enough time to catch up, and thus the head starter obtains a higher expected payoff than the latecomer does. When the deadline is long, the latecomer is highly likely to obtain a superior innovation than the head starter, and thus the latecomer obtains a higher expected payoff. In the latter case, the latecomer's initial apparent "disadvantage", in fact, puts it in a more favorable position than the head starter. When the deadline is sufficiently long, the head starter is doomed to lose the competition with a payoff of zero because of its unwillingness to search, and all benefits of the head start goes to the latecomer.

Then, does the result that the latecomer is in a more favorable position than the head starter when the deadline is long imply that the head start hurts the head starter and benefits the latecomer in the long run? Focus on the case in which the latecomer does not have an initial innovation. When the search cost is relatively low, the head start, in fact, always benefits the head starter, but the benefit ceases as the deadline extends. It also benefits the latecomer when the deadline is long. When the search cost is relatively high, the head start could potentially hurt the head starter.

If the head start is large, neither firm will conduct a search, because the latecomer is deterred from competition. In this scenario, no innovation or technological progress is created, and the head starter wins the contest directly. If the head start is small, both firms play the same equilibrium strategy as they do when neither firm has an initial innovation. In both cases, the head start benefits the head starter and hurts the latecomer.

Section 5.1 extends our model to include stages at which the firms sequentially have an option to discard their initial innovation before the contest starts. Suppose that both firms' initial innovations are of values in the middle range and that the deadline is long. If the head starter can take the first move in the game, it can increase its expected payoff by discarding its initial innovation and committing to search. When search cost is low, by sacrificing the initial innovation, the original head starter actually makes the competitor the new head starter; this new head starter has no incentive to discard its initial innovation or to search any more. It is possible that by discarding the head start, the original head starter may benefit both firms. When search cost is high, discarding the initial innovation is a credible threat to the latecomer, who will find the apparent leveling of the playing field discouraging to conducting a high-cost search. As a result, the head starter suppresses the innovation progress.

In markets, some firms indeed give up head starts (Ulhøi, 2004), and our result provides a partial explanation of this phenomenon. For example, Tesla gave up its patents for its advanced technologies on electric vehicles at an early stage of its business.[2] While there may be many reasons for doing so, one significant reason is to maintain Tesla's position as a leading innovator in the electronic vehicle market.[3] As Elon Musk (2014), the CEO of Tesla, wrote,

> technology leadership is not defined by patents, which history has repeatedly shown to be small protection indeed against a determined competitor, but rather by the ability of a company to attract and motivate the world's most talented engineers.[4]

[2]Toyota also gave up patents for its hydrogen fuel cell vehicles at an early stage.

[3]Another reason is to help the market grow faster by the diffusion of its technologies. A larger market increases demand and lowers cost.

[4]See "All Our Patent Are Belong To You," June 12, 2014, on http://www.teslamotors.com/blog/all-our-patent-are-belong-you.

Whilst Tesla keeps innovating to win a large share of the future market, its smaller competitors have less incentive to innovate since they can directly adopt Tesla's technologies. One conjecture which coincides with our result is that "Tesla might be planning to distinguish itself from the competitors it helps . . . by inventing and patenting better electric cars than are available today" (*Discovery Newsletter*, June 13, 2014).

Section 5.2 considers intermediate information disclosure. Suppose the firms are required to reveal their discoveries at an early time point after the starting of the contest, how would firms compete against each other? If the head start is in the middle range, before the revelation point, the head starter will conduct a search, whereas the latecomer will not. If the head starter obtains a very good innovation before that point, the latecomer will be deterred from competition. Otherwise, the head starter is still almost certain to lose the competition. Hence, such an information revelation at an early time point increases both the expected payoff to the head starter and the expected value of the winning innovation.

Section 6 compares the effects of a head start to those of a cost advantage and points out a significant difference. A cost advantage reliably encourages a firm to search more actively for innovations, whereas it discourages the firm's competitor.

Section 7 concludes the paper. The overarching message our paper conveys is that a market regulator who cares about long-run competitions in markets may not need to worry too much about the power of the current market dominating firms if these firms are not in excessively high positions. In the long run, these firms are to be defeated by latecomers. On the other hand, if the dominating firms are in excessively high positions, which deters entry, a regulator can intervene the market.

Literature

There is a large literature on innovation contests. Most work considers reduced form models (Fullerton and McAfee, 1999; Moldovanu and Sela, 2001; Baye and Hoppe, 2003; Che and Gale, 2003).[5] Head starts are studied in various forms of all-pay auctions. Leininger (1991), Konrad (2002), and Konrad and Leininger (2007) model a head start as a first-mover advantage in a sequential all-pay auction and study the first-mover's performance. Casas-Arce and Martinez-Jerez (2011), Siegel (2014b), and Seel (2014) model a head start as a handicap in a simultaneous all-pay auction and study the effect on the head starter. Kirkegard (2012) and Seel and Wasser (2014) also model a head start as a handicap in a simultaneous all-pay auction but study the effect on the auctioneer's expected revenue. Segev and Sela (2014) analyzes the effect a handicap

[5] Also see, for example, Hillman and Reiley (1989), Baye et al. (1996), Krishna and Morgan (1998), Che and Gale (1998), Cohen and Sela (2007), Schöttner (2008), Bos (2012), Siegel (2009, 2010, 2014a), Kaplan et al. (2003), and Erkal and Xiao (2015).

on the first mover in a sequential all-pay auction. Unlike these papers, we consider a framework in which players' decisions are dynamic.

The literature considering settings with dynamic decisions is scarce, and most studies focus on symmetric players. The study by Taylor (1995) is the most prominent.[6] In his symmetric T-period private search model, there is a unique equilibrium in which players continue searching for innovations until they discover one with a value above a certain cut-off. We extend Taylor's model to analyze the effects of a head start and find the long-run effects of the head start, which is our main contribution.

Seel and Strack (2013, 2015) and Lang et al. (2014) also consider models with dynamic decisions. Same as in our model, in these models each player also solves an optimal stopping problem. However, the objectives and the results of these papers are different from ours. In Seel and Strack (2013 2015), each player decides when to stop a privately observed Brownian motion with a drift. In Seel and Strack (2013), there is no deadline and no search cost and a process is forced to stop when it hits zero. They find that players do not stop their processes immediately even if the drift is negative. In Seel and Strack (2015), each search incurs a cost that depends on the stopping time. This more recent study finds that when noise vanishes the equilibrium outcome converges to the symmetric equilibrium outcome of an all-pay auction. Lang et al (2014) consider a multi-period model in which each player decides when to stop a privately observed stochastic points-accumulation process. They find that in equilibrium the distribution over successes converges to the symmetric equilibrium distribution of an all-pay auction when the deadline is long.

Our paper also contributes to the literature on information disclosure in innovation contests. Aoyagi (2010), Ederer (2010), Goltsman and Mukherjee (2011), Wirtz (2013) study how much information on intermediate performances a contest designer should disclose to the contestants. Unlike our model, these papers consider two-stage games in which the value of a contestant's innovation is its total outputs from the two stages. Halac et al. (2015) and Bimpikis et al. (2014) study the problem of designing innovation contests, which includes both the award structures and the information disclosure policies. Halac et al. (2015) consider a model in which each contestant searches for innovations, but search outcomes are binary. A contest ends after the occurrence of a single breakthrough, and a contestant becomes more and more pessimistic over time if there has been no breakthrough. Bimpikis et al. (2014) consider a model which shares some features with Halac et al.'s (2015). In the model, an innovation happens only if two breakthroughs are achieved by the contestants, the designer decides whether to disclose the information on whether the first breakthrough has been achieved by a contestant, and intermediate awards can be used. In both models, contestants are sym-

[6]Innovation contests were modeled as a race in which the first to reach a defined finishing line gains a prize, e.g., Loury (1979), Lee and Wilde (1980), and Reinganum (1981, 1982).

metric. In contrast, the contestants in our model are always asymmetric. Rieck (2010) studies information disclosure in a two-period model of Taylor (1995). In contrast to our finding, he shows that the contest designer prefers concealing the outcome in the first stage. Unlike all the above papers, Gill (2008), Yildirim (2005), and Akcigit and Liu (2014) address the incentives for contestants, rather than the designer, to disclose intermediate outcomes.

Last but most importantly, this paper contributes to the literature on the relationship between market structure and incentive for R&D investment. The debate over the effect of market structure on R&D investment dates back to Schumpeter (1934, 1942).[7] Due to the complexity of the R&D process, earlier theoretical studies tend to focus on one facet of the process. Gilbert and Newbery (1982), Fudenberg et al.(1983), Harris and Vickers (1985a, 1985b, 1987), Judd (2003), Grossman and Shapiro (1987), and Lippman and McCardle (1987) study preemption games. In these models, an incumbent monopolist has more incentive to invest in R&D than a potential entrant. In fact, a potential entrant sees little chance to win the competition, because of a lag at the starting point of the competition, and is deterred from competition. In our model, the intuition for the result in the case of a large head start is similar to this "preemption effect", except that no firm invests in our case.

By contrast, Arrow (1962) and Reinganum (1983, 1985) show, in their respective models, that an incumbent monopolist has less incentive to innovate than a new entrant.[8] The cause for this is what is called the "replacement effect" by Tirole (1997). While an incumbent monopolist can increase its profit by innovating, it has to lose the profit from the old technology once it adopts a new technology. This effectively reduces the net value of the new technology to the incumbent. It is then natural that a firm who has a lower value of an innovation has less incentive to innovate, which is exactly what happens in our model with asymmetric costs. On the other hand, our main result, on medium-sized head start, has an intuition very similar to the "replacement effect". Rather than a reduction in the value of an innovation to the head starter, a head start decreases the increase in the probability of winning from innovating. In both Reinganum's models and our model, an incumbent could have a lower probability of winning than a new entrant. However, different from her models, in our model an incumbent (head starter) can also have a lower expected payoff than a new entrant (latecomer).

[7] See Gilbert (2006) for a comprehensive survey.

[8] Doraszelski (2003) generalizes the models of Reinganum (1981, 1982) to a history-dependent innovation process model and shows, in some circumstances, the catching-up behavior in equilibrium.

1 Model

Firms and Tasks

There are two risk neutral firms, Firm 1 and Firm 2, competing for a prespecified prize, normalized to 1, in the contest. Time is continuous, and each firm searches for innovations before a deadline T. At the deadline T, each firm releases to the public the best innovation it has discovered, and the firm who releases a superior innovation wins the prize. If no firm has discovered any innovation, the prize is retained. If there is a tie between the two firms, the prize is randomly allocated to them with equal probability.

At any time point $t \in [0, T)$ before the deadline, each firm decides whether to continue searching for innovations. If a firm continues searching, the arrival of innovations in this firm follows a Poisson process with an arrival rate of λ. That is, the probability of discovering m innovations in an interval of length δ is $\frac{e^{-\lambda\delta}(\lambda\delta)^m}{m!}$. The values of innovations are drawn independently from a distribution F, defined on $(0, 1]$ with $F(0) := \lim_{a \to 0} F(a) = 0$. F is continuous and strictly increasing over the domain.

Each firm's search cost is $c > 0$ per unit of time. We assume that $c < \lambda$, because if $c > \lambda$ the cost is so high that no firm is going to conduct a search. To illustrate this claim, suppose Firm 2 does not search, Firm 1 will not continue searching if it has an innovation with a value above 0, whereas Firm 1's instantaneous gain from searching at any moment when it has no innovation is

$$\lim_{\delta \to 0} \frac{\sum_{m=1}^{+\infty} \frac{e^{-\lambda\delta}(\lambda\delta)^m}{m!} - c\delta}{\delta} = \lambda - c,$$

which is negative if $c > \lambda$.

Information

The search processes of the two firms are independent and with recall. Whether the opponent firm is actively searching is unobservable; whether a firm has discovered any innovation, as well as the values of discovered innovations, is private information until the deadline T.

For convenience, we say a firm is in a **state** $a \in [0, 1]$ at time t if the value of the best innovation it has discovered by time t is a, where $a = 0$ means that the firm has no innovation. The **initial states** of Firm 1 and Firm 2 are denoted by a_1^I and a_2^I, respectively. Firms' initial states are commonly known.

Strategies

In our model, each firm's information on its opponent is not updated. Hence, the game is static, although the firms' decisions are dynamic. In accordance with the standard result from search theory that each firm's optimal strategy is a constant cut-off rule, we make the following assumption.[9]

Assumption 1.1. *Firm i's strategy space is* $\mathcal{S}_i^F := \{-1\} \cup [a_i^I, \ 1]$.

A strategy $\hat{a}_i \in [a_i^I, \ 1]$ represents a **constant cut-off rule** of Firm i: at any time point $t \in [0, T)$, Firm i stops searching if it is in a state above $\hat{a}_i$ and continues searching if in a state below or at $\hat{a}_i$.[10] A strategy $\hat{a}_i = -1$ represents that Firm i does not conduct a search.

Suppose both firms have no initial innovation. Without this assumption, for any given strategy played by a firm's opponent, there is a constant cut-off rule being the firm's best response. Such a cut-off value being above zero is the unique best response strategy, ignoring elements associated with zero probability events. However, in the cases in which a firm is indifferent between continuing searching and not if it is in state 0, this firm has (uncountably) many best response strategies. The above assumption helps us to focus on the two most natural strategies: not to search at all and to search with 0 as the cut-off.[11] A full justification for this assumption is provided in the appendix.

Let $\tilde{P}[a|\hat{a}_i, a_i^I]$ denote the probability of Firm i ending up in a state below a if it adopts a strategy $\hat{a}_i$ and its initial state is a_i^I; let $E[cost|\hat{a}_i]$ denote Firm i's expected cost on search if it adopts a strategy $\hat{a}_i$. Firm i's ex ante expected utility is

$$U_i = \int_0^1 P[a|\hat{a}_{-i}, a_{-i}^I] dP[a|\hat{a}_i, a_i^I] - E[cost|\hat{a}_i].$$

Now, we are ready to study equilibrium behavior. The solution concept we use is Nash equilibrium. Before solving the case with a head starter, we first look at the case with no initial innovation.

2 The Symmetric-Firms Benchmark ($a_i^I = 0$)

In this section, we look at the benchmark case, in which both firms start with no innovation. It is in the spirit of Taylor's (1995), except that it is in continuous time.

[9]See Lippman and McCall (1976) for the discussion on optimal stopping strategies for searching with finite horizon and recall.

[10]Once Firm i stops searching at some time point it shall not search again later.

[11]Without this assumption, there can be additional best response strategies of the following type: a firm randomizes between searching and not searching until a time $T' < T$ with cutoff 0 and stops at T' even if no discovery was made.

The equilibrium strategies are presented below.[12]

Theorem 2.1. *Suppose* $a_1^I = a_2^I = 0$.

i. If $c \in [\frac{1}{2}\lambda(1+e^{-\lambda T}), \lambda)$, *there are two equilibria, in each of which one firm searches with* 0 *as the cut-off and the other firm does not search.*

ii. If $c \in [\frac{1}{2}\lambda(1 - e^{-\lambda T}), \frac{1}{2}\lambda(1 + e^{-\lambda T}))$, *there is a unique equilibrium, in which both firms search with* 0 *as the cut-off.*

iii. If $c \in (0, \frac{1}{2}\lambda(1 - e^{-\lambda T}))$, *there is a unique equilibrium, in which both firms search with* a^* *as the cut-off, where* $a^* > 0$ *is the unique value that satisfies*

$$\frac{1}{2}\lambda[1 - F(a^*)]\left[1 - e^{-\lambda T[1-F(a^*)]}\right] = c. \tag{1}$$

Proof. See Appendix A.2. □

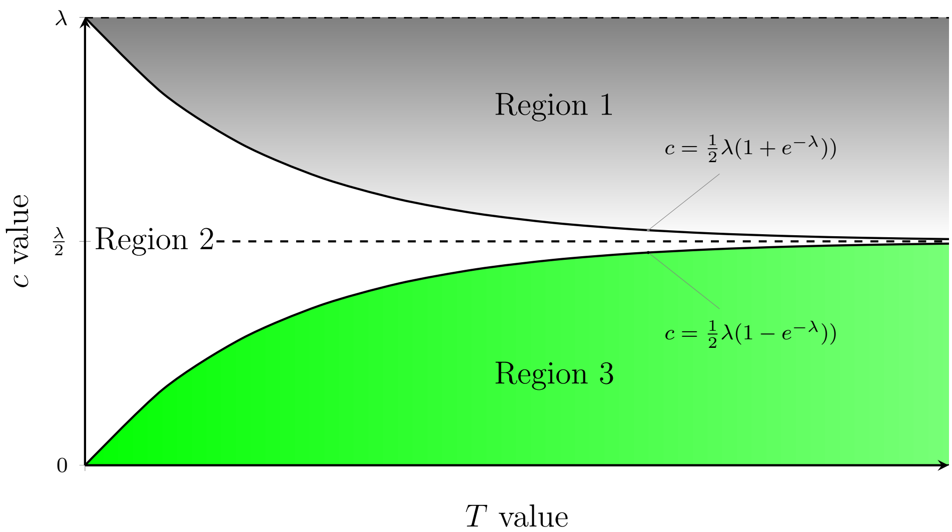

Figure 1

The result is illustrated in figure 1. The deadline-cost space is divided into three regions.[13] In Region 1, the search cost is so high that it is not profitable for both firms to innovate. In Region 2, both firms would like to conduct a search in order to discover an innovation with any value, but none has the incentive to spend additional effort to find an innovation with a high value. In Region 3, both firms exert efforts to find an innovation with a value above a certain level. In this case, a firm in the cut-off state is indifferent between continuing and stopping searching. This is represented by

[12] When search cost is low, the equilibrium is unique even without assumption 1.1. When search is high, without Assumption 1.1, there are additional symmetric equilibria of the following type: each firm randomizes between not participating and participating until a time $T' < T$ with 0 as the cutoff.

[13] In this paper, an " area" is defined by the interior of the corresponding area.

equation (1), in which $1-e^{-\lambda T[1-F(a^*)]}$ is the probability of a firm's opponent ending up in a state above a^* and $\frac{1}{2}[1-F(a^*)]$ is the increase in the probability of winning if the firm, in state a^*, obtains a new innovation. Hence, this equation represents that, in the cut-off state, the instantaneous increase in the probability of winning from continuing searching equals the instantaneous cost of searching. As T goes to infinity, $c = \frac{\lambda}{2}$ becomes the separation line for Case [i] and Case [iii].

Generally, there is no closed form solution for the cut-off value in Case [iii]. However, if the search cost is very low, we have a simple approximation for it.

Corollary 2.1. *Suppose $a_1^I = a_2^I = 0$. When c is small, $a^* \approx F^{-1}\left(1-\sqrt{\frac{2c}{\lambda^2 T}}\right)$.*

Proof. First, we assume that $\lambda T[1-F(a^*)]$ is small, and we come back to check that it is implied by that c is small. Applying equation (1), we have

$$\begin{aligned}
&\frac{c}{\lambda} = \frac{1}{2}[1-F(a^*)]\left[1-e^{-\lambda T[1-F(a^*)]}\right] \approx \frac{1}{2}\lambda T[1-F(a^*)]^2 \\
\Leftrightarrow &[1-F(a^*)]^2 \approx \frac{2c}{\lambda^2 T} \\
\Leftrightarrow &a^* \approx F^{-1}\left(1-\sqrt{\frac{2c}{\lambda^2 T}}\right) \quad \text{and} \quad \lambda T[1-F(a^*)] \approx \sqrt{2cT}.
\end{aligned}$$

□

For later reference we, based on the previous theorem, define a function $a^* : (0,\lambda) \times [0,+\infty) \to [0,1]$ where

$$a^*(c,T) = \begin{cases} 0 & \text{for } c \in [\frac{1}{2}\lambda(1-e^{-\lambda T}), \lambda) \\ \text{the } a^* \text{ that solves (1)} & \text{for } c \in (0, \frac{1}{2}\lambda(1-e^{-\lambda T})). \end{cases}$$

A simple property which will be used in later sections is stated below.

Lemma 2.1. *In Region 3, $a^*(c,T)$ is strictly increasing in T (and λ) and strictly decreasing in c.*

There are two observations. One is that $a^*(c,T) = 0$ if $c \geq \frac{\lambda}{2}$. The other is that $a^*(c,T)$ converges to $F^{-1}(1-\frac{2c}{\lambda})$ as T goes to infinity if $c < \frac{\lambda}{2}$, which derives from taking the limit of equation (1) w.r.t. T. Let us denote a_L^* as the limit of $a^*(c,T)$ w.r.t. T:

$$a_L^* := \lim_{T\to+\infty} a^*(c,T) = \begin{cases} 0 & \text{for } c \geq \frac{\lambda}{2}, \\ F^{-1}(1-\frac{2c}{\lambda}) & \text{for } c < \frac{\lambda}{2}. \end{cases}$$

We end this section by presenting a full rent dissipation property of the contest when the deadline approaches infinity.

Lemma 2.2. *Suppose $a_1^I = a_2^I = 0$. If $c < \frac{\lambda}{2}$, each firm's expected payoff in equilibrium goes to 0 as the deadline T goes to infinity.*

Proof. See Appendix A.2. □

The intuition is as follows. The instantaneous increase in the expected payoff from searching for a firm who is in state $a^*(c,T)$, the value of the equilibrium cut-off, is 0 (it is indifferent between continuing searching and not). If the deadline is finite, a firm in a state below $a^*(c,T)$ has a positive probability of winning even if it stops searching. Hence, the firms have positive rents in the contest. As the deadline approaches infinity, there is no difference between being in a state below $a^*(c,T)$ and at $a^*(c,T)$, because the firm will lose the contest for sure if it does not search. In either case the instantaneous increase in the expected payoff from searching is 0. Hence, in the limit the firms' rents in the contest are fully dissipated.

Though the equilibrium expected payoff goes to 0 in the limit, it is not monotonically decreasing to 0 as the deadline approaches infinity, because each firm's expected payoff converges to 0 as the deadline approaches 0 as well.[14]

3 Main Results: Exogenous Head Starts ($a_1^I > a_2^I$)

In this section, we add head starts into the study. Without loss of generality, we assume that Firm 1 has a better initial innovation than does Firm 2 before competition begins, i.e., $a_1^I > a_2^I$. We first derive the equilibrium strategies, and then we explore equilibrium properties.

3.1 Equilibrium Strategies

Theorem 3.1. *Suppose $a_1^I > a_2^I$.*

1. *For $a_1^I > F^{-1}(1-\frac{c}{\lambda})$, there is a unique equilibrium, in which no firm searches, and thus Firm 1 wins the prize.*

2. *For $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, there are many equilibria. In one equilibrium, both firms do not search. In the other equilibria, Firm 1 does not search and Firm 2 searches with a value $\hat{a}_2 \in [a_2^I, a_1^I]$ as the cut-off.*

3. *For $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, there is a unique equilibrium, in which Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off .*

[14] In fact, by taking the derivative of (12) (as in the appendix) w.r.t. T, one can show that the derivative at $T=0$ is $\lambda - c > 0$ and that, if $c < \frac{\lambda}{2}$, (12) is increasing in T for $T < \min\{\frac{1}{\lambda}\ln\frac{\lambda}{c}, \frac{1}{\lambda}\ln\frac{\lambda}{\lambda-2c}\}$ and decreasing in T for $T > \max\{\frac{1}{\lambda}\ln\frac{\lambda}{c}, \frac{1}{\lambda}\ln\frac{\lambda}{\lambda-2c}\}$.

4. *For $a_1^I = a^*(c,T)$, there are two equilibria. In one equilibrium, both firms search with a_1^I as the cut-off. In the other equilibrium, Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off.*

5. *For $a_1^I \in (0, a^*(c,T))$, there is a unique equilibrium, in which both firms search with $a^*(c,T)$ as the cut-off.*

Proof. See Appendix A.3. □

Remark. *Case [4] and [5] exist only when $c \leq \frac{1}{2}\lambda[1-e^{-\lambda T}]$ (Region 3).*

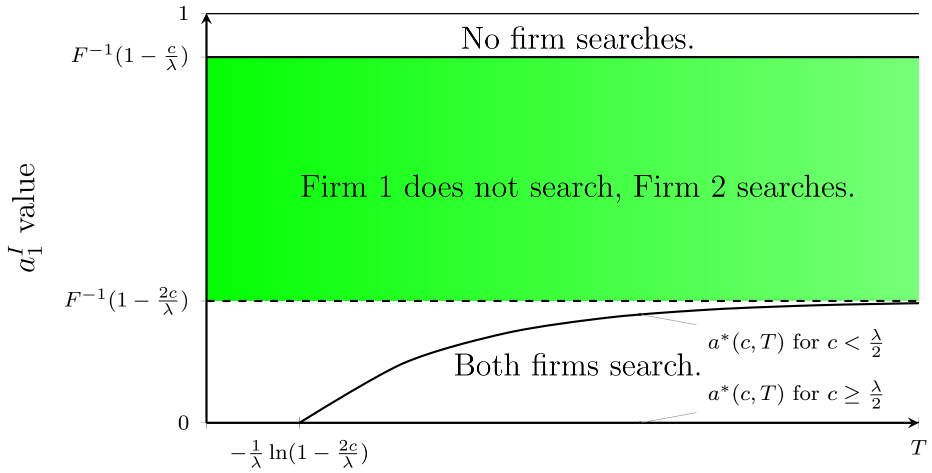

Figure 2: Thresholds (when $c < \frac{\lambda}{2}(1-e^{-\lambda T})$).

The thresholds in the theorem are depicted in figure 2. The leading case is Case [3], when the head start is in the middle range. A head start reduces the return of a search, in terms of the increase in the probability of winning. Having a sufficiently large initial innovation, Firm 1 loses incentive to search because the marginal increase in the probability of winning from searching for Firm 1 is too small compared to the marginal cost of searching, whether Firm 2 searches or not. Firm 2 takes advantage of that and commits to search until it discovers an innovation better than Firm 1's initial innovation. Hence, compared to its equilibrium behavior in the benchmark case, Firm 2 is more active in searching (in terms of a higher cut-off value) when Firm 1 has a medium-sized head start, and a larger value of head start forces Firm 2 to search more actively.

In Case [1], Firm 1's head start is so large that Firm 2 is deterred from competition because Firm 2 has little chance to win if it searches. Firm 1 wins the prize without incurring any cost. Moreover, it is independent of the deadline T.

In Case [5], in which Firm 1's head start is small, the head start has no effect on either firm's equilibrium strategy, and both firms search with $a^*(c,T)$ as the cut-off, same as in the benchmark case. The only effect of the head start is an increase in Firm 1's probability of winning (and a decrease in Firm 2's).

In brief, a comparison of Theorem 3.1 and Theorem 2.1 shows that a head start of Firm 1 does not alter its own equilibrium behavior but Firm 2's. The effect on Firm 2's equilibrium strategy is not monotone in the head start of Firm 1. The initial state of Firm 2, the latecomer, is irrelevant to the equilibrium strategies. Figure 3 illustrates how each firm's equilibrium strategy changes as the value of the initial innovation of Firm 1, the head starter, varies.

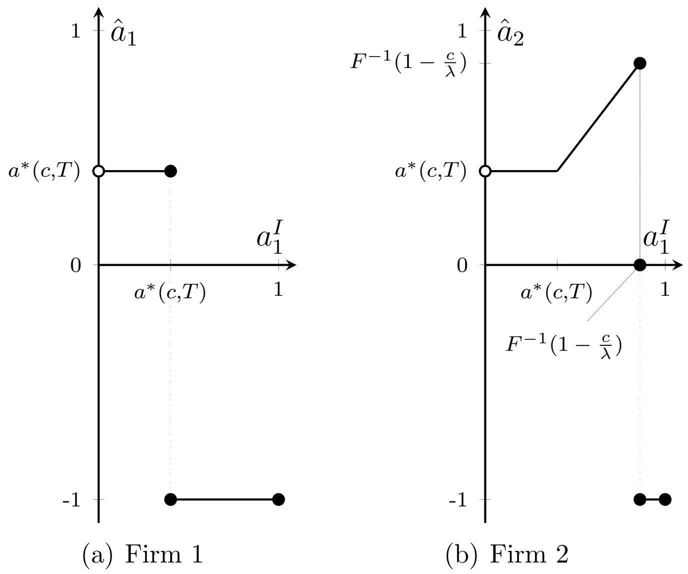

Figure 3: Firms' equilibrium cut-off values as the value of Firm 1's initial innovation, a_1^I, varies.

Figure 4 illustrates Firm 2'the best responses (when it has no initial innovation) to Firm 1's strategies for various values of Firm 1's initial states. The case in which Firm 1 has a high-value initial innovation is significantly different from the case in which Firm 1 has no initial innovation.

Turning back to Case [3] in the previous result, we notice that the lower bound for this case to happen does not converge to the upper bound as the deadline approaches infinity, i.e., $a_L^* < F^{-1}(1-\frac{c}{\lambda})$. The simplest but most interesting result of our paper, the case of "head starts and doomed losers", derives.

Corollary 3.1. *Suppose* $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$.

1. *Firm 2's (Firm 1's) probability of winning increases (decreases) in the deadline.*

2. *As* T *goes to infinity, Firm 2's probability of winning goes to* 1*, and Firm 1's goes to* 0*.*

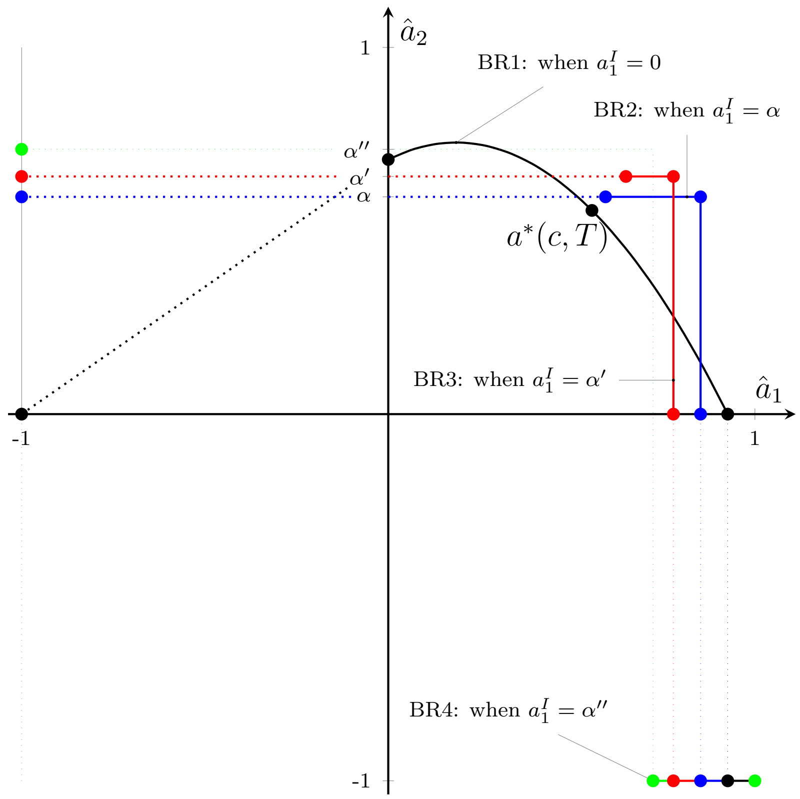

Figure 4: Best response projections for Firm 2 as a_1^I takes values in $\{-1, \alpha, \alpha', \alpha''\}$ ($c < \frac{\lambda}{2}(1 - e^{-\lambda T})$ and $a_2^I = 0$).

BR1 represents Firm 2's best responses when no firm has an initial innovation. If Firm 1 does not search, Firm 2 would search with 0 as the cut-off. If Firm 1 searches with 0 as the cut-off, Firm 2 would search with a cut-off higher than the equilibrium cut-off. As Firm 1 further rises its cut-off, Firm 2 would first rise its cut-off and then lower its cut-off. When the deadline is long, Firm 2 would not search if Firm 1's cut-off is high. BR2 and BR3 represent Firm 2's best responses when Firm 1 has an initial innovation with a value slightly above $a^*(c, T)$, the equilibrium cut-off when there is no initial innovation. In this case, if Firm 1 does not search, Firm 2's best response is to search with the value Firm 1's initial innovation as the cut-off. If Firm 1 searches with a cut-off slightly above the value of its initial innovation, Firm 2's best response is still to search with the value of Firm 1's initial innovation as the cut-off. Once Firm 1's cut-off is greater than a certain value, Firm 2 would not search. BR4 represents Firm 2's best responses when Firm 1 has a high-value initial innovation. In this case, if Firm 1 does not search, Firm 2 would still search with the value of Firm 1's initial innovation as the cut-off; if Firm 1 searches, Firm 2 would have no incentive to search. On the other hand, when the value of Firm 1's initial innovation is above $a^*(c, T)$, Firm 1's best response to any strategy of Firm 2 is not to search.

Proof. In equilibrium Firm 1 does not search and Firm 2 searches with a_1^I as the cut-off. Firm 2's probability of winning is thus

$$1 - e^{-\lambda T[1-F(a_1^I)]},$$

which is increasing in T, and it converges to 1 as T goes infinity. Firm 1's probability of winning is $e^{-\lambda T[1-F(a_1^I)]}$, which is, in the contrast, decreasing in T, and it converges to 0 as T approaches infinity. $\square$

This property results from our assumption that search processes are with recall. The larger the head start is, the smaller the marginal increase in the probability of winning from searching is, given any strategy played by the latecomer. Hence, even if the head starter knows that in the long run the latecomer will almost surely obtain an innovation with a value higher than its initial innovation, the head starter is not going to conduct a search as long as the instantaneous increase in the probability of winning is smaller than the instantaneous cost of searching.

3.2 Payoff Comparison across Firms

A natural question arises: which firm does a head start favor? Will Firm 1 or Firm 2 achieve a higher expected payoff? To determine that, we need a direct comparison of the two firms' expected payoffs. When $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, the difference between the payoffs of Firm 1 and Firm 2 is[15]

$$D^F(T, a_1^I) := e^{-\lambda T[1-F(a_1^I)]} - (1 - e^{-\lambda T[1-F(a_1^I)]})(1 - \frac{c}{\lambda[1-F(a_1^I)]}). \tag{2}$$

The head start of Firm 1 favors Firm 1 (Firm 2) if $D^F(T, a_1^I) > (<)0$.

$D^F(T, a_1^I)$ is increasing in a_1^I and decreasing in T. Hence, a longer deadline tends toward to favor Firm 2 when the head start is in the middle range. Since

$$D^F(0, a_1^I) = 1 > 0$$

and

$$\lim_{T\to\infty} D^F(T, a_1^I) = -(1 - \frac{c}{\lambda[1-F(a_1^I)]}) < 0 \text{ for any } a^I < F^{-1}(1 - \frac{c}{\lambda}),$$

there must be a unique $\hat{T}(a_1^I) > 0$ such that $DE(\hat{T}(a_1^I), a_1^I) = 0$. The following result derives.

[15] $1 - e^{-\lambda T[1-F(a_1^I)]}$ is Firm 2's probability of obtaining an innovation better than Firm 1's initial innovation, a_1^I, and $\frac{1}{\lambda[1-F(a_1^I)]}$ is the unconditional expected interarrival time of innovations with a value higher than a_1^I. The second term in $D^F(T, a_1^I)$ thus represents the expected payoff of Firm 2.

Proposition 3.1. *For $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, there is a unique $\hat{T}(a_1^I) > 0$ such that Firm 1 (Firm 2) obtains a higher expected payoff if $T < (>) \hat{T}(a_1^I)$.*

That is, for any given value of the head start in the middle range $(a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, the head start favors the latecomer (head starter) if the deadline is long (short). The effects of a head start do not vanish as the deadline approaches infinity. In fact, as the deadline approaches infinity, the head start eventually pushes the whole share of the surplus to Firm 2.

Lemma 3.1. *As the deadline increases to infinity,*

1. *for $a_1^I \in (0, a_L^*)$, both Firms' equilibrium payoffs converge to 0;*
2. *for $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, Firm 1's equilibrium payoff converges to 0, whereas Firm 2's equilibrium payoff converges to $1 - \frac{c}{\lambda[1-F(a_1^I)]} \in (0, \frac{1}{2})$.*

Proof. [1] follows from Lemma 2.2. [2] follows from Corollary 3.1 and the limit of Firm 2's expected payoff

$$(1 - e^{-\lambda T[1-F(a^I)]}) \left(1 - \frac{c}{\lambda[1-F(a_1^I)]}\right) \tag{3}$$

w.r.t. T. □

A comparison to Lemma 2.2 shows that, just as having no initial innovation, when Firm 1 has an innovation whose value is not of very high, its expected payoff still converges to 0 as the deadline becomes excessively long. When there is no initial innovation, the expected total surplus for the firms (i.e., the sum of the two firms' expected payoff) converges to 0. In contrast, when there is a head start with a value above a_L^*, the expected total surplus is strictly positive even when the deadline approaches infinity. However, as it approaches infinity, this total surplus created by the head start of Firm 1 goes entirely to Firm 2, the latecomer, if the head start is in the middle range. The intuition is as follows. For Firm 1, it is clear that its probability of winning converges to 0 as the deadline goes to infinity. For Firm 2, we first look at the case that $a_1^I = F^{-1}(1-\frac{c}{\lambda})$. In this case Firm 2 is indifferent between searching and not searching, and thus the expected payoff is 0. As the deadline approaches infinity, both expected cost of searching and the probability of winning converges to 1, if Firm 2 conducts a search. Then, if a_1^I is below $F^{-1}(1-\frac{c}{\lambda})$ (but above a_L^*), as the deadline approaches infinity, Firm 2's probability of winning still goes to 1, but the expected cost of searching drops to a value below 1 because it adopts a lower cut-off for stopping. Hence, Firm 2's expected payoff converges to a positive value.

The relationship between the rank order of the two Firms' payoffs and the deadline is illustrated in figure 5, in each of which Firm 2 obtains a higher expected payoff at each point in the colored area. (a) is for the cases in which $c \leq \frac{\lambda}{2}$. In these cases a longer deadline tends to favor the latecomer. (b) and (c) are for the cases in which $c > \frac{\lambda}{2}$. In these cases, the rank order is not generally monotone in the deadline and the head start.

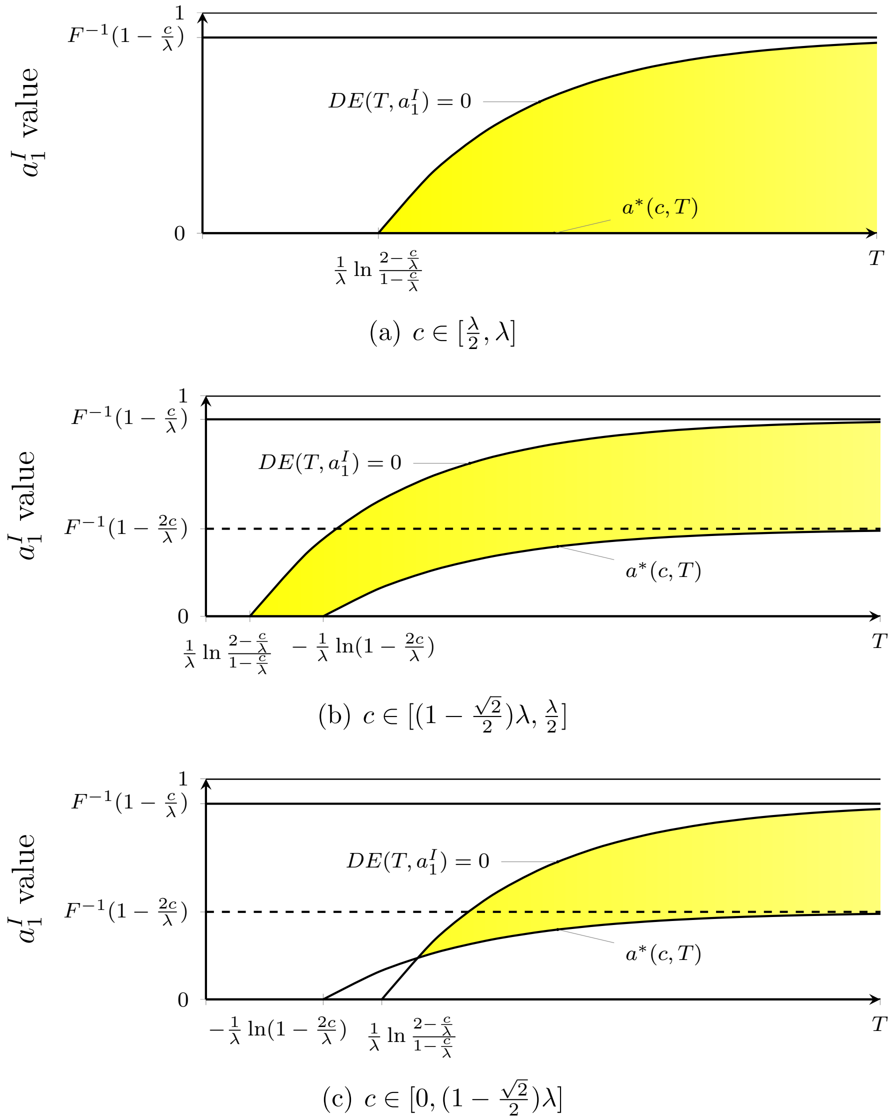

Figure 5: When (a_1^I, T) lies in the colored area, the head start favors the latecomer.

We notice in all the figures that if the deadline is sufficiently short, a head start is ensured to bias toward Firm 1, whereas if it is long, only a relatively large head start biases toward Firm 1.

4 Effects of Head Starts on Payoffs

In this section, we study the effects of a head start on both firms' payoffs. Suppose Firm 2 has no initial innovation, who does a head start of Firm 1 benefit or hurt? The previous comparison between Theorem 2.1 and Theorem 3.1 already shows that a head start a_1^I benefits Firm 1 and hurts Firm 2 if $a_1^I < a^*(c,T)$ or $a_1^I > F^{-1}(1-\frac{c}{\lambda})$. In the former case, which happens only when $a^*(c,T) > 0$ (Region 3 of figure 1), both firms search with $a^*(c,T)$ as the cut-off, the same as when there is no head start, and the head start increases Firm 1's probability of winning and decreases Firm 2's. As the deadline goes to infinity, the expected payoffs to both firms converge to 0, with the effect of the head start disappearing. In the latter case, Firm 1 always obtains a payoff of 1, and Firm 2 always 0.

The interesting case occurs then when the head start is in the middle range, $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, which is the focus in the remaining parts of the paper. To answer the above question regarding the head start being in the middle range, we first analyze the case that point (c,T) lies in Regions 2 and 3 (in figure 1), and then we turn to analyze the case of Region 1.

4.1 Regions 2 and 3

In the previous section, we showed that for T being sufficiently long, Firm 1 is almost surely going to lose the competition if a_1^I is in the middle range. Although it seems reasonable that in this case a head start may make Firm 1 worse off, the following proposition shows that this conjecture is not true.

Proposition 4.1. *Suppose $a_2^I = 0$. In Regions 2 and 3, in which $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$, a head start $a_1^I > 0$ always benefits Firm 1, compared to the equilibrium payoff it gets in the benchmark case.*

To give the intuition, we consider the case of $a^*(c,T) > 0$. Suppose Firm 1 has a head start $a_1^I = a^*(c,T)$. As shown in Case [4] of Theorem 3.1, we have the following two equilibria: in one equilibrium both firms search with $a^*(c,T)$ as the cut-off; in another equilibrium Firm 1 does not search and Firm 2 searches with $a^*(c,T)$ as the cut-off. Firm 1 is indifferent between these two equilibria, hence its expected payoffs from both equilibria are $e^{-\lambda T[1-F(a^*(c,T))]}$, the probability of Firm 2 finding no innovation with a value higher than $a^*(c,T)$. However, Firm 1's probability of winning increases in its head start, hence a larger head start gives Firm 1 a higher expected payoff.

The above result itself corresponds to expectation. What unexpected is the mechanism through which Firm 1 gets better off. As a head start gives Firm 1 a higher position, we would expect that it is better off by (1) having a better chance to win and (2) spending less on searching. Together with Theorem 3.1, the above proposition shows that Firm 1 is better off purely from an increase in the probability of winning when $a_1^I < a^*(c,T)$; purely from spending nothing on searching when $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ (though there could be a loss from a decrease in the probability of winning); from an increase in the probability of winning and a reduction in the cost of searching when $a_1^I > F^{-1}(1-\frac{c}{\lambda})$.

In contrast to the effect of a head start of Firm 1 on Firm 1's own expected payoff, the effect on Firm 2's expected payoff is not clear-cut. Instead of giving a general picture of the effect, we present some properties in the following.

Proposition 4.2. *Suppose* $a_2^I = 0$.

1. *A head start* $a_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$ *hurts Firm 2 if the deadline* T *is sufficiently small.*
2. *If* $c < \frac{\lambda}{2}$, *a head start* $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$ *benefits Firm 2 if the deadline is sufficiently long.*

Proof. See Appendix A.3. □

Case [1] occurs because a head start of Firm 1 reduces Firm 2's probability of winning and may increase its expected cost of searching. Case [2] follows from Propositions 4.1 and 3.1. Because a head start of Firm 1 always benefits Firm 1 and a long deadline favors Firm 2, a head start must also benefit Firm 2 if the deadline is long.[16]

Figure 6 illustrates how Firm 2's equilibrium payoff changes as Firm 1's head start increases. In particular, a head start of Firm 1 slightly above $a^*(c,T)$, the equilibrium cut-off when there is no initial innovation, benefits Firm 2 if $a^*(c,T)$ is low. Some more conditions under which a head start benefits or hurts the latecomer are given below.

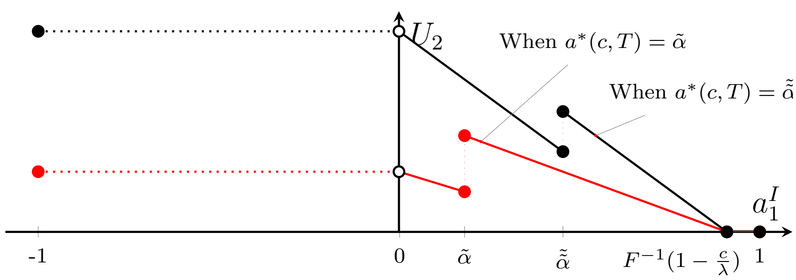

Figure 6: Firm 2's equilibrium payoffs as $a^*(c,T)$ varies.

Proposition 4.3. *In Region 2 and 3, in which* $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$,

[16]Alternatively, it also follows from Lemmas 2.2 and 3.1. If the deadline is very long and the head start of Firm 1 is in the middle range, Firm 2's payoff converges to 0 same as in the benchmark case and some positive value in head start case.

1. *if*

$$(1 - e^{-\lambda T[1-F(a^*(c,T))]}) - \frac{1}{2}(1 - e^{-2\lambda T}) > 0, \tag{4}$$

there exists a $\tilde{a}_1^I \in (a^*(c,T), F^{-1}(1 - \frac{c}{\lambda}))$ *such that the head start* a_1^I *hurts Firm* 2 *if* $a_1^I \in (\tilde{a}_1^I, F^{-1}(1 - \frac{c}{\lambda}))$ *and benefits Firm* 2 *if* $a_1^I \in (a^*(c,T), \tilde{a}_1^I)$*;*

2. *if* (4) *holds in the opposite direction, any head start* $a_1^I \in (a^*(c,T), F^{-1}(1 - \frac{c}{\lambda}))$ *hurts Firm* 2*.*

Proof. See Appendix A.3. □

The first term on the left side of inequality (4) is Firm 2's probability of winning in the equilibrium in which Firm 2 searches and Firm 1 does not search in the limiting case that Firm 1 has a head start of $a^*(c,T)$. The second term, excluding the minus sign, is Firm 1's probability of winning when there is no head start. The expected searching costs are the same in both cases. The following corollary shows some scenarios in which inequality (4) holds.

Corollary 4.1. *In Region 2, when* $a^*(c,T) = 0$*, inequality* (4) *holds.*

This shows that for search cost lying in the middle range, a head start of Firm 1 must benefit Firm 2, if it is slightly above 0. The simple intuition is as follows. When Firm 1 has such a small head start, Firm 2's cut-off value of searching increases by only a little bit, and thus the expected cost of searching also increases slightly. However, the increase in Firm 2's probability of winning is very large, because Firm 1, when having a head start, does not search any more. Thus, in this case Firm 2 is strictly better off.

Lastly, even though Firm 1 does not search when the head start $a_1^I > a^*(c,T)$, it seems that a low search cost may benefit Firm 2. On the contrary, a head start of Firm 1 would always hurt Firm 2 when the search cost is sufficiently small.

Corollary 4.2. *For any fixed deadline* T*, if the search cost is sufficiently small, inequality* (4) *holds in the opposite direction.*

Proof. As c being close to 0, $a^*(c,T)$ is close to 1, and thus the term on left side of inequality (4) is close to $-\frac{1}{2}(1 - e^{-2\lambda T}) < 0$. □

That is because when c is close to 0, $a^*(c,T)$ is close to 1, and the interval in which Firm 1 does not search while Firm 2 searches is very small, and thus the chance for Firm 2 to win is too low when $a_1^I > a^*(c,T)$, even though the expected cost of searching is low as well.

4.2 Region 1

Since there are multiple equilibria in the benchmark case when (c, T) lies in Region 1, whether a head start hurts or benefits a firm depends on which equilibrium we compare to. If we compare the two equilibria in each of which Firm 1 does not search and Firm 2 searches, then the head start benefits Firm 1 and hurts Firm 2. If we compare to the other equilibrium in the benchmark case, the outcome is not clear-cut.

Proposition 4.4. *Suppose $a_2^I = 0$. In Region 1, in which $c > \frac{1}{2}\lambda(1 + e^{-\lambda T})$ and $a^*(c, T) = 0$, for $a_1^I \in (0, F^{-1}(1 - \frac{c}{\lambda}))$, if*

$$(1 - e^{-\lambda T})(1 - \frac{c}{\lambda}) - e^{-\lambda T[1 - F(a_1^I)]} < 0, \tag{5}$$

Firm 1's equilibrium payoff is higher than its expected payoff in any equilibrium in the benchmark case. If the inequality holds in the opposite direction, Firm 1's equilibrium payoff is lower than its payoff in the equilibrium in which Firm 1 searches and Firm 2 does not search in the benchmark case.

This result is straightforward. The first term on the left side of inequality (5) is Firm 1's expected payoff in the equilibrium in which Firm 1 searches and Firm 2 does not in the benchmark case and the second term, excluding the minus sign, is its expected payoff when there is no head start.

Moreover, the left hand side of inequality (5) strictly increases in T, and it reaches -1 when T approaches 0 and $1 - \frac{c}{\lambda}$ when T approaches infinity. The intermediate value theorem insures that inequality (5) holds in the opposite direction for the deadline T being large.

As a result of the above property, when the head start is small and the deadline is long, in an extended game in which Firm 1 can publicly discard its head start before the contest starts, there are two subgame perfect equilibria: in one equilibrium, Firm 1 does not discard its head start and Firm 2 searches with the Firm 1's initial innovation value as the cut-off; in the other equilibrium, Firm 1 discards the head start and searches with 0 as the cut-off and Firm 2 does not search. Hence, there is the possibility that Firm 1 can improve its expected payoff if it discards its head start.

Last, we discuss Firm 2's expected payoff. The result is also straightforward.

Proposition 4.5. *Suppose $a_2^I = 0$. In Region 1, in which $c > \frac{1}{2}\lambda(1 + e^{-\lambda T})$, for $a_1^I \in (0, F^{-1}(1 - \frac{c}{\lambda}))$, Firm 2's equilibrium payoff is*

- *less than its expected payoff in the equilibrium in which Firm 1 does not search and Firm 2 searches in the benchmark case, and*
- *higher than the payoff in the equilibrium in which Firm 1 searches and Firm 2 does not search in the benchmark case.*

Proof. Compared to the equilibrium in which Firm 2 searches in the benchmark case, in the equilibrium when Firm 1 has a head start, Firm 2 has a lower expected probability of winning and a higher expected cost because of a higher cut-off, and and thus a lower expected payoff. But this payoff is positive. □

5 Extended Dynamic Models

In this section, we study two extended models.

5.1 Endogenous Head Starts ($a_1^I > a_2^I > 0$)

We first study how the firms would play if each firm has the option to discard its initial innovation before the contest starts. Formally, a game proceeds as below.

Model*:

- Stage 1: Firm i decides whether to discard its initial innovation.
- Stage 2: Firm i's opponent decides whether to discard its initial innovation.
- Stage 3: Upon observing the outcomes in the previous stages, both firms simultaneously start playing the contest as described before.

The incentive for a head starter to discard its initial innovation when the latecomer has no initial innovation has been studied in the previous section. The focus of the section is on the case in which both firms have an initial innovation in the middle range.[17] The main result of this section is as follows.

Proposition 5.1. *Suppose $a_1^I, a_2^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$. In Model* with Firm 1 having the first move, there is a $\breve{T}(a_1^I, a_2^I) > 0$ such that*

- *if $T > \breve{T}(a_1^I, a_2^I)$, there is a unique subgame perfect equilibrium (SPE), in which Firm 1 discards its initial innovation and searches with a_2^I as the cut-off and Firm 2 keeps its initial innovation and does not search;*
- *if $T < \breve{T}(a_1^I, a_2^I)$, subgame perfect equilibria exist, and in each equilibrium Firm 2 searches with a_1^I as the cut-off and Firm 1 keeps its initial innovation and does not search.*

Proof. See Appendix A.4. □

[17] As shown in Proposition 4.1, when (c, T) lies in Regions 2 and 3, the head starter with a medium-sized initial innovation has no incentive to discard its head start if the latecomer has a no innovation. The head starter would also have no incentive to discard its initial innovation when the latecomer has a low-value initial innovation.

This proposition shows that in the prescribed scenario Firm 1 is better off giving up its initial innovation if the deadline is long.[18] The intuition is simple. For the deadline being long, Firm 1's expected payoff is low, because its probability of winning is low. By giving up its initial innovation, it makes Firm 2 the head starter, and thus Firm 1 obtains a higher expected payoff than before by committing to searching whereas Firm 2 has no incentive to search. Yet the reasoning for the case in which $c \leq \frac{\lambda}{2}$ differs from that for the case in which $c > \frac{\lambda}{2}$. After Firm 1 discards its initial innovation, Firm 2 turns to the new head start firm. In the former case, Firm 2 would then have no incentive to discard its initial innovation any more as shown in Proposition 4.1, and its dominant strategy in the subgame is not to search whether Firm 1 is to search or not. In the latter case, Firm 2 may have the incentive to discard its initial innovation and search if the deadline is long. However, discarding the initial innovation is a credible threat for Firm 1 to deter Firm 2 from doing that.

Remark. *When the deadline is sufficiently long, by giving up the initial innovation, Firm 1 makes itself better off but Firm 2 worse off. However, if the deadline is not too long, by doing so, Firm 1 can benefit both firms. This is because the total expected cost of searching after Firm 1 discards its initial innovation is lower than before and hence there is an increase in the total surplus for the two firms. It is then possible that*

[18]Discarding a head start is one way to give up one's initial leading position. In reality, a more practical and credible way is to give away the head start innovation. A head start firm could give away it patent for its technology. By doing so, any firm can use this technology for free. That is, every firm's initial state becomes a_1^I. If firms can enter the competition freely, the value of the head start technology is approximately zero to any single firm, because everyone has approximately zero probability to win with this freely obtained innovation. For a head start being in the middle range, the market is not large enough to accommodate two firms to compete. Hence, to model giving away head starts with free entry to the competition, we can study a competition between two firms but with some modified prize allocation rules. Formally, the game proceeds as below.

Model**:

- Stage 1: Firm 1 decides whether to give away its initial innovation.
- Stage 2: Upon observing the stage 1 outcome, Firm 1 and Firm 2 simultaneously start playing the contest as described before.
- If Firm 1 gives away its initial innovation:
  - Both firms' states at time 0 become a_1^I.
  - The prize is retained if no firm is in a state above a_1^I at the deadline T.
  - The firm with a higher state, which is higher than a_1^I, at the deadline wins.
- If Firm 1 retains its initial innovation, the firm with a higher state at the deadline wins.

Suppose $a_1^I \in (a_L^*, F^{-1}(1 - \frac{c}{\lambda}))$. If the deadline is long, in Model** there are two subgame perfect equilibria. In one equilibrium, Firm 1 gives away its initial innovation and searches with a_1^I as the cut-off and Firm 2 does not search. In the other equilibrium, Firm 1 retains its initial innovation and does not search and Firm 2 searches with a_1^I as the cut-off. However, forward induction selects the first equilibrium as the refined equilibrium, because giving away a head start is a credible signal of Firm 1 to commit to search.

both firms get a share of the increase in the surplus. We illustrate that in the following example.

Example 5.1. *Suppose F is the uniform distribution, $c = \frac{1}{3}$, $\lambda = 1$, $a_1^I = \frac{1}{2}$, and $a_2^I = \frac{1}{3}$. If Firm 1 discards its initial innovation, then its expected payoff would be $\frac{1}{2}(1 - e^{-\frac{T}{3}})$, and Firm 2's expected payoff would be $e^{-\frac{T}{3}}$; if Firm 1 does not discard its initial innovation, then its expected payoff would be $e^{-\frac{T}{2}}$, and Firm 2's expected payoff would be $\frac{1}{3}(1 - e^{-\frac{T}{2}})$.*

Firm 1 would be better off by discarding its initial innovation if $T > 2.52$. If $T \in (2.52, 3.78)$, by discarding the initial innovation, Firm 1 makes both firms better off. If T is larger, then doing so would only make Firm 2 worse off.

The previous result is conditional on Firm 1 having the first move. If Firm 2 has the first move, it may, by discarding its initial innovation, be able to prevent Firm 1 from discarding its own head start and committing to searching. However, if the deadline is not sufficiently long, Firm 1 would still have the incentive to discard its initial innovation.

Proposition 5.2. *Suppose $a_1^I, a_2^I \in (a_L^*, F^{-1}(1 - \frac{c}{\lambda}))$. In Model* with Firm 2 having the first move, there is a $\hat{T}(a_1^I, a_2^I) > 0$ such that*

- *for $T > \hat{T}(a_1^I, a_2^I)$, there is a unique SPE, in which Firm 2 discards its initial innovation and searches with a_2^I as the cut-off and Firm 1 keeps the initial innovation and does not search;*
- *for $T \in (\breve{T}(a_1^I, a_2^I), \hat{T}(a_1^I, a_2^I))$, there is a unique SPE, in which Firm 2 keeps its initial innovation and does not search and Firm 1 discards its initial innovation and searches with a_1^I as cut-off;*
- *for $T < \breve{T}(a_1^I, a_2^I)$, subgame perfect equilibria exist, and in each equilibrium Firm 2 searches with cut-off a_1^I and Firm 1 keeps its initial innovation and does not search.*

Proof. See Appendix A.4. □

In the middle range of the deadline, even though Firm 2 can credibly commit to searching and scare Firm 1 away from competition by discarding its initial innovation, it is not willing to do so, yet Firm 1 would like to discard its initial innovation and commit to searching. This is because Firm 1's initial innovation is of a higher value than Firm 2's. The cut-off value of the deadline at which Firm 2 is indifferent between discarding the initial innovation to commit to searching, and keeping the initial innovation, is higher than that of Firm 1.

5.2 Intermediate Information Disclosure

In the software industry, it is common to preannounce with a long lag to launch (Bayus et al., 2001). Many firms do that by describing the expected features or demonstrating prototypes at trade shows. Many other firms publish their findings in a commercial disclosure service, such as Research Disclosure, or in research journals.[19] Suppose there is a regulator who would like to impose an intermediate information disclosure requirement on innovation contests. What are the effects of the requirement on firms' competition strategies and the expected value of the winning innovation. Specifically, suppose there is a time point $t_0 \in (0, T)$ at which both firms have to reveal everything they have, how would firms compete against each other?

When the head start a_1^I is larger the threshold $F^{-1}(1-\frac{c}{\lambda})$, it is clear that no firm has an incentive to conduct any search. When the head start is below this threshold, if t_0 is very close to T, information revelation has little effect on the firms' strategies. Both firms will play approximately the same actions before time t_0 as they do when there is no revelation requirement. After time t_0, the firm in a higher state at time t_0 stops searching. The other firm searches with this higher state as the cut-off if this higher state is below $F^{-1}(1-\frac{c}{\lambda})$, and stops searching as well if it is higher than $F^{-1}(1-\frac{c}{\lambda})$.

Our main finding in this part regards the cases in which the head start is in the middle range and the deadline is sufficiently far from the information revelation point.[20] That is, firms have to reveal their progress at an early stage of a competition.

Proposition 5.3. *Suppose at a time point $t_0 \in (0, T)$ both firms have to reveal their discoveries. For $a_1^I \in (a_L^*, F^{-1}(1-\frac{c}{\lambda}))$ and $a_2^I < a_1^I$, if $T - t_0$ is sufficiently large, there is a unique subgame perfect equilibrium, in which*

- *Firm 1 searches with $F^{-1}(1-\frac{c}{\lambda})$ as the cut-off before time t_0 and stops searching from time t_0;*
- *Firm 2 does not search before time t_0 and searches with a_1^I as the cut-off from time t_0 if $a_1^I < F^{-1}(1-\frac{c}{\lambda})$.*

Proof. See Appendix A.4. □

In the proof, we show that between time 0 and time t_0, the dominant action of Firm 2 is not to search, given that the equilibria in the subgames from time t_0 are described as in Theorem 3.1. If Firm 1 is in a state higher than the threshold $F^{-1}(1-\frac{c}{\lambda})$ at time t_0, Firm 2's effort will be futile if it searches before time t_0. If Firm 1 is in a state in

[19]Over 90% of the world's leading companies have published disclosures in Research Disclosure's pages (see www.researchdisclosure.com).

[20]Generally, for the cases in which $a_1^I < a_L^*$, there are many subgame perfect equilibria, including two equilibria in each of which one firm searches with $F^{-1}(1-\frac{c}{\lambda})$ as the cut-off between time 0 and time t_0 and the other firm does not.

between its initial state a_1^I and and the threshold $F^{-1}(1-\frac{c}{\lambda})$ at time t_0, Firm 2 has the chance to get into a state above the threshold $F^{-1}(1-\frac{c}{\lambda})$ and thus a continuation payoff of 1, but this instantaneous benefit only compensates the instantaneous cost of searching. Firm 2 also has the chance to get into a state above that of Firm 1 but below the threshold $F^{-1}(1-\frac{c}{\lambda})$ at time t_0, which results in a continuation payoff of approximately 0 if the deadline is sufficiently long, whereas it obtains a strictly positive payoff if it does not search before time t_0. It is thus not worthwhile for Firm 2 to conduct a search before time t_0. If Firm 2 does not search before time t_0, Firm 1 then has the incentive to conduct a search if the deadline is far from t_0. If it does not search, it obtains a payoff of approximately 0 when $T-t_0$ is sufficiently large. If it conducts a search before time t_0, the benefit from getting into a higher state can compensate the cost.

An early stage revelation requirement therefore hurts the latecomer and benefits the head starter. It gives the head starter a chance to get a high-value innovation so as to deter the latecomer from competition. It also increases the expected value of the winning innovation.

6 Discussion: Asymmetric Costs ($a_i^I = 0$, $c_1 < c_2$)

In this section, we show that, compared to the effects of head starts, the effects of cost advantages are simpler. A head start probably discourages a firm from conducting searching and can either discourage its competitor from searching or encourage its competitor to search more actively. In contrast, a cost advantage encourages a firm to search more actively and discourages its opponent.

We now assume that the value of pre-specified prize to Firm i, $i=1,2$, is V_i and that the search cost is for Firm i is C_i per unit of time. However, at each time point Firm i only makes a binary decision on whether to stop searching or to continue searching. Whether it is profitable to continue searching depends on the ratio of $\frac{C_i}{V_i}$ rather than the scale of V_i and C_i. Therefore, we can normalize the valuation of each player to be 1 and the search cost to be $\frac{C_i}{V_i} =: c_i$. W.l.o.g, we assume $c_1 < c_2$. For convenience, we define a function

$$I(a_i|a_j, c_i) := \lambda \int_{a_i}^{\bar{a}} [Z(a|a_j) - Z(a_i|a_j)]dF(a) - c_i,$$

where $Z(a|a_j)$ is defined, in Lemma A.2 in the appendix, as Firm j's probability of ending up in a state **below** a if it searches with a_j as the cut-off. We emphasize on the most important case, in which both firms' search costs are low.

Proposition 6.1. *For $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$, there must exist a unique equilibrium*

(a_1^*, a_2^*), *in which* $a_1^* > a_2^* \geq 0$. *Specifically,*

1. *if* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, *the unique equilibrium is a pair of cut-off rules* (a_1^*, a_2^*), $a_1^* > a_2^* > 0$, *that satisfy*

$$\lambda \int_{a_i^*}^{\bar{a}} \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)\right] dF(a) = c_i;$$

2. *if* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) \leq 0$, *the unique equilibrium is a pair of cut-off rules* $\left(F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$.

Proof. See Appendix A.5 □

The existence of equilibrium is proved by using Brouwer's fixed point theorem. As expected, a cost (valuation) advantage would drive a firm to search more actively than its opponent. The following statement shows that while an increase in cost advantage of the firm in advantage would make the firm more active in searching and its opponent less active, a further cost disadvantage of the firm in disadvantage would make both firms less active in searching.

Proposition 6.2. *For* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, *in which case there is a unique equilibrium* (a_1^*, a_2^*), $a_1^*, a_2^* > 0$,

1. *for fixed* c_2, $\frac{\partial a_1^*}{\partial c_1} < 0$ *and* $\frac{\partial a_2^*}{\partial c_1} > 0$;
2. *for fixed* c_1, $\frac{\partial a_1^*}{\partial c_2} < 0$ *and* $\frac{\partial a_2^*}{\partial c_2} < 0$.

Proof. See Appendix A.5 □

The intuition is simple. When the cost of the firm in advantage decreases, this firm would be more willing to search, while the opponent firm would be discouraged because the marginal increase in the probability of winning from continuing searching in any state is reduced, and therefore the opponent firm would lower its cut-off. When the cost of the firm at a disadvantage increases, the firm would be less willing to search, and the opponent firm would consider it less necessary to search actively because the probability of winning in any state has increased.

A comparison between the equilibrium strategies in this model and that of the benchmark model can be made.

Corollary 6.1. *For* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1-\sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, *in which case there is a unique equilibrium* (a_1^*, a_2^*), $a_1^* > a_2^* > 0$, a_1^* *and* a_2^* *satisfy*

1. $a_1^* < a^*(c, T)$ *for the corresponding* $c = c_2 > c_1$;

$c_2 \setminus c_1$	Region 1	Region 2	Region 3
Region 1	(a_1^*, a_2^*)	$\left(F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$	$(0, -1)$
Region 2	/	$(0, 0)$	$(0, -1)$
Region 3	/	/	$(0, -1), (-1, 0)$

Table 1: Equilibria in all non-marginal cases.

2. $a_2^* < a^*(c, T)$ *for the corresponding* $c = c_1 < c_2$*.*

Based on the benchmark model, a cost reduction for Firm 1 will result in both firms searching with cut-offs below the original one; a cost increase for Firm 2 will certainly result in Form 2 searching with a cut-off below the original one.

The equilibrium for the other non-marginal cases (conditional on $c_1 < c_2$), together with the above case, are stated in Table 1 without proof. The regions in Table 1 are the same as in Figure 1. The row (column) number indicates in which region c_1 (c_2) lies, and each element in each cell represents a corresponding equilibrium. For example, the element in the cell at the second row and the second column means that for $c_1, c_2 \in (\frac{1}{2}\lambda(1-e^{-\lambda T}), \frac{1}{2}\lambda(1+e^{-\lambda T}))$, there is a unique equilibrium, in which both firms search with cut-off 0. This shows that Firm 1 is more active in searching than is Firm 2.

Remark. *Similar results can be found in a model with the same search cost but with different arrival rates of innovations for the two firms.*

7 Concluding Remarks

This paper has studied the long-run effects of head starts in innovation contests in which each firm decides when to stop a privately observed repeated sampling process before a preset deadline. Unlike an advantage in innovation cost or innovation ability, which encourages a firm to search more actively for innovations and discourages its opponent, a head start has non-monotone effects. The head starter is discouraged from searching if the head start is large, and its strategy remains the same if the head start is small. The latecomer is discouraged from searching if the head start is large but is encouraged to search more actively if it is in the middle range. Our main finding is that, if the head start is in the middle range, in the long run, the head starter is doomed to lose the competition with a payoff of zero and the latecomer will take the entire surplus for the competing firms. As a consequence, our model can exhibit either the "preemption effect" or the "replacement effect", depending on the value of the head start.

Our results have implications on antitrust problems. Market regulators have concerns that the existence of market dominating firms, such as Google, may hinder competitions, and they take measures to curb the monopoly power of these companies. For instance, the European Union voted to split Google into smaller companies.[21] Our results imply that in many cases the positions of the dominant firms are precarious. In the long run, they will be knocked off their perch. These firms' current high positions, in fact, may promote competitions in the long run because they encourage their rivals to exert efforts to innovate and reach high targets. Curbing the power of the current dominating firms may benefit the society and these firms' rivals in the short run, but in the long run it hurts the society because it discourages innovation. However, the the dominating firms' positions are excessively high, which deters new firms from entering the the market, a market regulator could take some actions.

The results have also implications on R&D policies. When selecting an R&D policy, policy makers have to consider both the nature of the R&D projects and the market structure. If the projects are on radical innovations, subsidizing innovation costs effectively increases competition when the market is blank (no advanced substitutive technology exists in the market). However, when there is a current market dominating firm with an existing advanced technology, a subsidy may not be effective. The dominating firm has no incentive to innovate, and the latecomer, even if it is subsidized, will not innovate more actively.

In our model we have only one head starter and one latecomer. The model can be extended to include more than two firms, and similar results still hold. One extension of our paper is to study the designing problem in our framework. For example, one question is how to set the deadline. If the designer is impatient, she may want to directly take the head starter's initial innovation without holding a contest; if she is patient, she may set a long deadline in order to obtain a better innovation. Some other extensions include: to consider a model with a stochastic number of firms; to consider a model with cumulative scores with or without regret instead of a model with repeated sampling.

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A Appendix

A.1 Preliminaries

To justify Assumption 1.1, we show in the following that, for any given strategy played by a firm's opponent, there is a constant cut-off rule being the firm's best response. If the cut-off value is above zero, it is actually the unique best response strategy, ignoring elements associated with zero probability events. We argue only for the case that both firms' initial states are 0. The arguments for the other cases are similar and thus are omitted.

Suppose $a_1^I = a_2^I = 0$. For a given strategy played by Firm j, we say at time t

$$\underline{a}_i^t := \inf\{\tilde{a} \in A | \text{Firm } i \text{ weakly prefers stopping to continuing searching in state } \tilde{a}\}$$

is Firm i's **lower optimal cut-off** and

$$\bar{a}_i^t := \inf\{\tilde{a} \in A | \text{Firm } i \text{ strictly prefers stopping to continuing searching in state } \tilde{a}\}$$

is Firm i's **upper optimal cut-off** .

Lemma A.1. *Suppose $a_1^I = a_2^I = 0$. For any fixed strategy played by Firm j, Firm i's best response belongs to one of the three cases.*

i. Not to search: $\bar{a}_i^t = \underline{a}_i^t = -1$ for all $t \in [0, T]$,

ii. Search with a constant cut-off rule $\hat{a}_i \geq 0$: $\bar{a}_i^t = \underline{a}_i^t = \hat{a}_i \geq 0$ for all $t \in [0, T]$.

iii. Both not to search and search until being in a state above 0: $\bar{a}_i^t = 0$ and $\underline{a}_i^t = -1$ for all $t \in [0, T)$.

Proof of Lemma A.1. Fix a strategy of Firm j. Let $P(a)$ denote the probability of Firm j ending up in a state **below** a at time T. $P(a)$ is either constant in a or strictly increasing in a. It is a constant if and only if Firm j does not search.[22] If this is the case, Firm i's best response is to continue searching with a fixed cut-off $\hat{a}_i^t = \bar{a}^{t_i} = \underline{a}^{t_i} = 0$ for all t. In the following, we study the case in which $P(a)$ is strictly increasing in a.

Step 1. We argue that, given a fixed strategy played by Firm j, Firm i's best response is a (potentially history-dependent) cut-off rule. Suppose at time t Firm i is in a state $\tilde{a} \in [0,1]$. If it is strictly marginally profitable to stop (continue) searching at t, then it is also strictly marginally profitable to continue searching if it is in a state higher (lower) than $\tilde{a}$. Let the upper and lower optimal cut-offs at time t be $\bar{a}_i^t$ and $\underline{a}_i^t$, respectively, as defined previously.

Step 2. We show that $\{\bar{a}_i^t\}_{t=0}^T$ and $\{\underline{a}_i^t\}_{t=0}^T$ should be history-independent. We use a discrete version to approximate the continuous version. Take any $\tilde{t} \in [0,T)$. Let $\{t_l\}_{l=0}^k$, where $t_l - t_{l-1} = \frac{T-\tilde{t}}{k} =: \delta$ for $l = 1, ..., k$, be a partition of the interval $[\tilde{t}, T]$. Suppose Firm i can only make decisions at $\{t_l\}_{l=0}^k$ in the interval $[\tilde{t}, T]$. Let $\{\bar{a}^{t_l}\}_{l=0}^{k-1}$ and $\{\underline{a}^{t_l}\}_{l=0}^{k-1}$ be the corresponding upper and lower optimal cut-offs, respectively, and $G^\delta(a)$ be Firm i's probability of discovering **no** innovation with a value **above** a in an interval δ.

At t_{k-1}, for Firm i in a state a, if it stops searching, the expected payoff is $P(a)$; if it continues searching, the expected payoff is

$$\begin{aligned} &G^\delta(a)P(a) + \int_a^1 P(\tilde{a})dG^\delta(\tilde{a}) - \delta c_i \\ =&P(a) + \int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c_i. \end{aligned}$$

The firm strictly prefers continuing searching if and only if searching in the last period strictly increases its expected payoff,

$$e^\delta(a) := \int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c_i > 0.$$

$e^\delta(a)$ strictly decreases in a and $e^\delta(1) \leq 0$. Because $e^\delta(0)$ can be either negative or positive, we have to discuss several cases.

Case 1. If $e^\delta(0) < 0$, Firm i is strictly better off stopping searching in any state $a \in [0,1]$. Thus, $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = -1$.

Case 2. If $e^\delta(0) = 0$, Firm i is indifferent between stopping searching and continuing searching with 0 as the cut-off, if it is in state 0; strictly prefers stopping searching, if it is in any state above 0. Then $\bar{a}^{t_{k-1}} = 0$ and $\underline{a}^{t_{k-1}} = -1$.

Case 3. If $e^\delta(0) > 0 \geq \lim_{a\to 0} e^\delta(a)$, Firm i is strictly better off continuing searching in state 0, but stopping searching once it is in a state above 0. Thus, $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = 0$.

Case 4. If $\lim_{a\to 0} e^\delta(a) > 0$, then Firm i's is strictly better off stopping searching if it

[22]More generally, it is constant if and only if the opponent firm conducts search with a measure 0 over $[0,T]$.

is in a state above $\hat{a}^{t_{k-1}}$ and continuing searching if it is in a state below $\hat{a}^{t_{k-1}}$, where the optimal cut-off $\hat{a}^{t_{k-1}} > 0$ is the unique value of a that satisfies,

$$\int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c = 0.$$

Thus, in this case $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = \hat{a}^{t_{k-1}}$.

Hence, the continuation payoff at $t_{k-1} \geq 0$ for Firm i in a state $a \in [0,1]$ is

$$\omega(a) = \begin{cases} P(a) + \int_a^1 [P(\tilde{a}) - P(a)]dG^\delta(\tilde{a}) - \delta c & for\ a < \underline{a}^{t_{k-1}} \\ P(a) & for\ a \geq \underline{a}^{t_{k-1}}. \end{cases}$$

Then, we look at the time point t_{k-2}. In the following, we argue that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$. The argument for $\underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}}$ is very similar and thus is omitted.

First, we show that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$. Suppose $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$. Suppose Firm i is in state $\bar{a}^{t_{k-2}}$ at time t_{k-2}. Suppose Firm i searches between t_{k-2} and t_{k-1}. If it does not discover any innovation with a value higher than $\bar{a}^{t_{k-2}}$, then at the end of this period it stops searching and takes $\bar{a}^{t_{k-2}}$. However, $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$ implies

$$\begin{aligned} 0 &= \int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})]dG^\delta(\tilde{a}) - \delta c_i \\ &< \int_{\bar{a}^{t_{k-1}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-1}})]dG^\delta(\tilde{a}) - \delta c \leq 0. \end{aligned}$$

The search cost is not compensated by the increase in the probability of winning from searching between t_{k-2} and t_{k-1}, and thus the firm strictly prefers stopping searching to continuing searching at time t_{k-2}, which contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Hence, it must be the case that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$.

Next, we show that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

In *Case 1*, it is straightforward that Firm i strictly prefers stopping searching at t_{k-2}, since it is for sure not going to search between t_{k-1} and t_k. Hence, Firm i stops searching before t_{k-1}, and $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}} = \underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}} = -1$.

For $\bar{a}^{t_{k-1}} \geq 0$, we prove by contradiction that $\bar{a}^{t_{k-2}} < \bar{a}^{t_{k-1}}$ is not possible. Suppose the inequality holds. If Firm i stops searching at t_{k-2}, it would choose to continue searching at t_{k-1}, and its expected continuation payoff at t_{k-2} is $\omega(\bar{a}^{t_{k-2}})$. If the firm continues searching, its expected continuation payoff is

$$\omega(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^1 [\omega(a) - \omega(\bar{a}^{t_{k-2}})]dG^\delta(a) - \delta c. \tag{6}$$

In *Cases 2 and 3*, $\bar{a}^{t_{k-1}} = 0$ implies $\bar{a}^{t_{k-2}} = -1$. Then,

$$\begin{aligned}
&\int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) - \delta c_i \\
=&\int_{-1}^{1} [P(a)] dG^{\delta}(a) - \delta c_i \\
=&e^{\delta}(-1) \\
\geq&0
\end{aligned}$$

which means that Firm i in state 0 is weakly better off continuing searching between t_{k-2} and t_{k-1}, which implies that $\bar{a}^{t_{k-2}} \geq 0$, resulting in a contradiction.

For *Case 4*, in which $\bar{a}^{t_{k-1}} > 0$, we have in (6)

$$\begin{aligned}
&\int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) \\
=&\int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\left(P(a) + \int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a})\right) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a})\right)\right] dG^{\delta}(a) \\
&\quad + \int_{\bar{a}^{t_{k-1}}}^{1} \left[P(a) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a})\right)\right] dG^{\delta}(a) \\
=&\int_{\bar{a}^{t_{k-2}}}^{1} \left[P(a) - P(\bar{a}^{t_{k-2}})\right] dG^{\delta}(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a})\right] dG^{\delta}(a) \\
&\quad - \int_{\bar{a}^{t_{k-2}}}^{1} \left[\int_{\bar{a}^{t_{k-2}}}^{1} [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a})\right] dG^{\delta}(a) \\
=&G^{\delta}(\bar{a}^{t_{k-2}}) \int_{\bar{a}^{t_{k-2}}}^{1} \left[P(a) - P(\bar{a}^{t_{k-2}})\right] dG^{\delta}(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_{a}^{1} [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a})\right] dG^{\delta}(a) \\
>&0.
\end{aligned}$$

Hence, at t_{k-2} Firm i would strictly prefer continuing searching, which again contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Consequently, $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

By backward induction from t_{k-1} to t_0, we have $\bar{a}^{t_0} = \bar{a}^{t_{k-1}}$. Taking the limit we get

$$\bar{a}^{t} = \lim_{\delta \to 0} \bar{a}^{T-\delta} =: \bar{a} \quad \text{for all } t \in [0, T).$$

Similarly,

$$\underline{a}^{t} = \lim_{\delta \to 0} \underline{a}^{T-\delta} =: \underline{a} \quad \text{for all } t \in [0, T).$$

In addition, $\bar{a} \neq \underline{a}$ when and only when $\bar{a} = 0$ and $\underline{a} = -1$.

As a consequence, Firm i's best response is not to search, if $\bar{a} = \underline{a} = -1$; to continue searching if it is in a state below $\bar{a}$ and to stop searching once the firm in a state above

$\bar{a}$, if $\bar{a} = \underline{a} \geq 0$. $\square$

In brief, the above property is proved by backward induction. Take Case [ii] for example. If at the last moment a firm is indifferent between continuing and stopping searching when it is in a certain state, which means the increase in the probability of winning from continuing searching equals the cost of searching, and therefore there is no gain from searching. Immediately before the last moment the firm should also be indifferent between continuing searching and not given the same state. This is because, if the firm reaches a higher state from continuing searching, it weakly prefers not to search at the last moment, and thus the increase in the probability of winning from continuing searching at this moment equals the cost of searching as well. By induction, the firm should be indifferent between continuing and stopping searching in the same state from the very beginning.

In Case [iii], Firm i generally has uncountably many best response strategies. By Assumption 1.1, we rule out most strategies and consider only two natural strategies: not to search at all and to search with 0 as the cut-off.

Lemma A.2. *Suppose a firm's initial state is* 0*, and she searches with a cut-off* $\hat{a} \geq 0$*. Then, the firm's probability of ending up in a state* ***below*** $a \in [0,1]$ *at time* T *is*

$$Z(a|\hat{a},T) = \begin{cases} 0 & if\ a = 0 \\ e^{-\lambda T[1-F(a)]} & if\ 0 < a \leq \hat{a} \\ e^{-\lambda T[1-F(\hat{a})]} + \left[1 - e^{-\lambda T[1-F(\hat{a})]}\right] \frac{F(a)-F(\hat{a})}{1-F(\hat{a})} & if\ a > \hat{a}. \end{cases}$$

$1 - e^{-\lambda T[1-F(\hat{a})]}$ is the probability that the firm stops searching before time T, and $\frac{F(a)-F(\hat{a})}{1-F(\hat{a})}$ is the conditional probability that the innovation above the threshold the firm discovers is in between $\hat{a}$ and a.

Proof of Lemma A.2. For $a = 0$, it is clear that $Z(a|\hat{a},T) = 0$.

For $0 < a \leq \hat{a}$,

$$Z(a|\hat{a},T) = \sum_{l=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^l}{l!} F^l(a) = e^{-\lambda T[1-F(a)]}.$$

For $a > \hat{a}$, we approximate it by a discrete time model. Let $\{t_l\}_{l=0}^{k}$, where $0 = t_0 < t_1 < ... < t_k = T$, be a partition of the interval $[0,T]$, and let $\delta_l = t_l - t_{l-1}$ for $l = 1, 2, ..., k$. Define π as

$$\parallel \pi \parallel = \max_{1 \leq l \leq k} |\delta_l|.$$

Then,

$$
\begin{aligned}
Z(a|\hat{a},T) =& Z(\hat{a}|\hat{a},T) + \lim_{\|\pi\|\to 0}\sum_{l=1}^{k} Z(a|\hat{a},t_{l-1})\left[\sum_{n=1}^{\infty}\frac{e^{-\lambda\delta_l}(\lambda\delta_l)^n}{n!}[F^n(a)-F^n(\hat{a})]\right] \\
=& Z(\hat{a}|\hat{a},T) + \lim_{\|\pi\|\to 0}\sum_{l=1}^{k} e^{-\lambda t_{l-1}[1-F(\hat{a})]}\lambda e^{-\lambda\delta_l}\big([F(a)-F(\hat{a})]+O(\delta_l)\big)\delta_l \\
=& Z(\hat{a}|\hat{a},T) + \int_0^T \lambda e^{-\lambda t[1-F(\hat{a})]}[F(a)-F(\hat{a})]dt \\
=& Z(\hat{a}|\hat{a},T) + \left[1-e^{-\lambda T[1-F(\hat{a})]}\right]\frac{F(a)-F(\hat{a})}{1-F(\hat{a})},
\end{aligned}
$$

where the second term on the right hand side of each equality is the firm's probability of ending up in a state between $\hat{a}$ and a. The term $Z(\hat{a}|\hat{a},t_n)$ used here is a convenient approximation when δ_l is small. The second equality is derived from the fact that

$$
\sum_{n=2}^{\infty}\frac{(\lambda\delta_l)^n}{n!}[F^n(a)-F^n(a^*)] < \frac{\lambda^2\delta_l^2}{2(1-\lambda\delta_l)} = o(\delta_l).
$$

□

Lemma A.3. *Given $a > a'$, $Z(a|\tilde{a},T) - Z(a'|\tilde{a},T)$*

1. *is constant in $\tilde{a}$ for $\tilde{a} \geq a$;*
2. *strictly decreases in $\tilde{a}$ for $\tilde{a} \in (a',a)$;*
3. *strictly increases in $\tilde{a}$ for $\tilde{a} \leq a'$.*

This single-peaked property says that the probability of ending up in a state between a' and a is maximized if a firm chooses strategy a'.

Proof of Lemma A.3. First, we show that $\frac{1-e^{-\lambda Tx}}{x}$ strictly decreases in x over $(0,1]$ as follows. Define $s := \lambda T$ and take x_1, x_2, $0 < x_1 < x_2 \leq 1$, we have

$$
\frac{1-e^{-sx_1}}{x_1} > \frac{1-e^{-sx_2}}{x_2},
$$

implied by

$$
\begin{aligned}
&\frac{\partial(1-e^{-sx_1})x_2-(1-e^{-sx_2})x_1}{\partial s} = x_1x_2(e^{-sx_1}-e^{-sx_2}) \geq 0 \quad (=0 \text{ iff } s=0) \quad \text{and} \\
&(1-e^{-sx_1})x_2-(1-e^{-sx_2})x_1 = 0 \quad \text{for} \quad s=0.
\end{aligned}
$$

Next, define $x := 1 - F(a)$, $x' := 1 - F(a')$, and $\tilde{x} := 1 - F(\tilde{a})$. We have

$$Z(a|\tilde{a}, T) - Z(a'|\tilde{a}, T) = \begin{cases} e^{-\lambda T x} - e^{-\lambda T x'} & \text{for } \tilde{a} \geq a \\ (1 - e^{-\lambda T x'}) - (1 - e^{-\lambda T \tilde{x}})\frac{x}{\tilde{x}} & \text{for } \tilde{a} \in (a', a) \\ (1 - e^{-\lambda T \tilde{x}})\frac{x' - x}{\tilde{x}} & \text{for } \tilde{a} \leq a'. \end{cases}$$

It is independent of $\tilde{a}$ for $\tilde{a} \geq a$, strictly increasing in $\tilde{x}$ and thus strictly decreasing in $\tilde{a}$ for $\tilde{a} \leq a'$, and strictly decreasing in $\tilde{x}$ and thus strictly increasing in $\tilde{a}$ for $\tilde{a} \leq a'$. □

Lemma A.4. *Suppose Firm j with initial state 0 plays a strategy $\hat{a}_j$. Then, the* ***instantaneous gain*** *on payoff from searching for Firm i in a state a is*

$$\lambda \int_{a_i}^{1} [Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)] dF(a) - c.$$

Proof of Lemma A.4. For convenience, denote $H(a)$ as $Z(a|\hat{a}_j, T)$ for short. The instantaneous gain from searching for Firm i in a state a_i is

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{\left(e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c\right) - H(a_i)}{\delta} \\
=&\lim_{\delta \to 0} \frac{-(1 - e^{-\lambda\delta})H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
=&\lim_{\delta \to 0} \frac{-\lambda\delta e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
=& - \lambda H(a_i) + \lambda \left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] - c \\
=&\lambda \int_{a_i}^1 \left[Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)\right] dF(a) - c.
\end{aligned}$$

□

A.2 Proofs for the Benchmark Case

Proof of Theorem 2.1. We prove the theorem case by case.

Case[i]. When Firm i does not search, Firm j's best response is to search with cut-off 0. For $\frac{1}{2}\lambda(1 + e^{-\lambda T}) \leq c$, when Firm j searches with any cut-off $a_j \geq 0$, Firm i's best response is not to search, since the instantaneous gain from searching for Firm i in state

0 is

$$\begin{aligned}
&\lambda \int_0^1 Z(a|a_j, T) dF(a) - c \\
\leq &\lambda \int_0^1 Z(a|0, T) dF(a) - c \\
= &\lambda \int_0^1 [e^{-\lambda T} + (1 - e^{-\lambda T}) F(a)] dF(a) - c \\
= &\lambda \left[e^{-\lambda T} + \frac{1}{2}(1 - e^{-\lambda T}) \right] - c \\
= &\frac{1}{2}\lambda(1 + e^{-\lambda T}) - c \\
\leq &0 \quad (= 0 \text{ iff } \frac{1}{2}\lambda(1 + e^{-\lambda T}) = c),
\end{aligned}$$

where the first inequality follows from Lemma A.3. Hence, there are two pure strategy equilibria, in each of which one firm does not search and the other firm searches with 0 as the cut-off, and if $\frac{1}{2}\lambda(1 + e^{-\lambda T}) = c$ there is also an equilibrium in which both firms search with 0 as the cut-off.

Case [ii]. First, we show that any strategy with a cut-off value higher than zero is a dominated strategy. When Firm j does not search, Firm i prefers searching with 0 as the cut-off to any other strategy. Suppose Firm j searches with $\hat{a}_j \geq 0$ as the cut-off. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\begin{aligned}
&\lambda \int_{a_i}^1 [Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)] dF(a) - c \\
\leq &\lambda \int_{a_i}^1 [Z(a|a_i, T) - Z(a_i|a_i, T)] dF(a) - c \\
= &\frac{1}{2}\lambda(1 - e^{-\lambda T})[1 - F(a_i)]^2 - c \\
< &0,
\end{aligned}$$

where the first inequality follows from Lemma A.3. Hence, once Firm i is in a state above 0, it has no incentive to continue searching any more. In this case, Firm i prefers either not to conduct any search or to search with 0 as the cut-off to any strategy with a cut-off value higher than zero.

Second, we show that the prescribed strategy profile is the unique equilibrium. It is sufficient to show that searching with 0 as the cut-off is the best response to searching with 0 as the cut-off. Suppose Firm j searches with 0 as the cut-off, the instantaneous gain from searching for Firm i in state $a = 0$ is

$$\begin{aligned}&\lambda \int_0^1 Z(a|0, T) dF(a) - c \\ =&\frac{1}{2}\lambda(1 + e^{-\lambda T}) - c \\ >&0.\end{aligned}$$

That is, Firm i is strictly better off continuing searching if it is in state 0, and strictly better off stopping searching once it is in a state above 0. Hence, the prescribed strategy profile is the unique equilibrium.

Case [iii]. First, we prove that among the strategy profiles in which each firm searches with a cut-off higher than 0, the prescribed symmetric strategy profile is the unique equilibrium. Suppose a pair of cut-off rules (a_1^*, a_2^*), in which $a_1^*, a_2^* > 0$, is an equilibrium, then Firm i in state a_i^* is indifferent between continuing searching and not. That is, by Lemma A.4, we have

$$\lambda \int_{a_i^*}^1 \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)\right] dF(a) - c = 0. \tag{7}$$

Suppose $a_1^* \neq a_2^*$. W.l.o.g., we assume $a_1^* < a_2^*$. Then,

$$\begin{aligned}c =&\lambda \int_{a_1^*}^1 [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) \\ >&\lambda \int_{a_2^*}^1 [Z(a|a_2^*, T) - Z(a_2^*|a_2^*, T)] dF(a) \\ >&\lambda \int_{a_2^*}^1 [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) = c\end{aligned}$$

resulting in a contradiction. Hence, it must be the case that $a_1^* = a_2^*$.

Next, we show the existence of equilibrium by deriving the unique equilibrium cut-off value $a^* := a_1^* = a_2^*$ explicitly. Applying Lemma A.2 to (7), we have

$$\begin{aligned}&\lambda \int_{a^*}^1 \left[1 - e^{-\lambda T[1-F(a^*)]}\right] \frac{F(a) - F(a^*)}{1 - F(a^*)} dF(a) = c \\ \Leftrightarrow&\frac{1}{2}[1 - F(a^*)] \left[1 - e^{-\lambda T[1-F(a^*)]}\right] = \frac{c}{\lambda}.\end{aligned} \tag{8}$$

The existence of a solution is ensured by the intermediate value theorem: when $F(a^*) = 1$, the term on the left hand side of (8) equals to 0, smaller than $\frac{c}{\lambda}$; when $F(a^*) = 0$, it equals to $\frac{1-e^{-\lambda T}}{2}$, larger than or equals to $\frac{c}{\lambda}$. The uniqueness of the solution is insured by that the term on the left hand side of the above equality is strictly decreasing in a^*.

Second, we show that there is no equilibrium in which one firm searches with 0 as the cut-off. Suppose Firm j searches with 0 as the cut-off. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\begin{aligned} &\lambda \int_{a_i}^{1} [Z(a|0,T) - Z(a_i|0,T)]dF(a) - c \\ =&\lambda \int_{a_i}^{1} (1-e^{-\lambda T})[F(a) - F(a_i)]dF(a) - c \\ =&\frac{1}{2}\lambda(1-e^{-\lambda T})[1-F(a_i)]^2 - c, \end{aligned} \tag{9}$$

which is positive when $a_i = 0$ and negative when $a_i = 1$. By the intermediate value theorem, there must be a value $\hat{a}_i > 0$ such that (9) equals 0 when $a_i = \hat{a}_i$. Hence, Firm i's best response is to search with $\hat{a}_i$ as the cut-off. However, if Firm i searches with $\hat{a}_i$ as the cut-off, it is not Firm j's best response to search with 0 as the cut-off, because

$$\begin{aligned} 0 =& \int_{\hat{a}_j}^{1} [Z(a|0,T) - Z(\hat{a}_j|0,T)]dF(a) - c \\ <& \int_{\hat{a}_j}^{1} [Z(a|\hat{a}_j,T) - Z(\hat{a}_j|\hat{a}_j,T)]dF(a) - c \\ <& \int_{0}^{1} [Z(a|\hat{a}_j,T) - Z(0|\hat{a}_j,T)]dF(a) - c, \end{aligned}$$

which means that Firm j strictly prefers continuing searching when it is in a state slightly above 0.

Last, we show that there is no equilibrium in which one firm does not search. Suppose Firm j does not search. Firm i's best response is to search with 0 as the cut-off. However, Firm j then strictly prefers searching when it is in state 0, since the instantaneous gain from searching for the firm in state 0 is again

$$\lambda \int_{0}^{1} Z(a|a_i,T)dF(a) - c > 0.$$

□

Proof of Lemma 2.2. The expected total cost of a firm who searches with a cut-off

$a \geq 0$ is

$$c\left[\int_0^T \frac{\partial(1-Z(a|a,t))}{\partial t} t dt + TZ(a|a,T)\right]$$
$$=(1-e^{-\lambda T[1-F(a)]})\frac{c}{\lambda[1-F(a)]}, \tag{10}$$

which strictly increases in a. In Regions 2 and 3, in equilibrium, the probability of winning for each firm is

$$\frac{1}{2}[1-Z^2(0|a^*(c,T),T)]$$
$$=\frac{1}{2}(1-e^{-2\lambda T}), \tag{11}$$

and thus the expected payoff to each firm is the difference between the expected probability of winning (11) and the expected search cost (10), setting a to be $a^*(a,T)$:

$$\frac{1}{2}(1-e^{-2\lambda T})-(1-e^{-\lambda T[1-F(a^*(c,T))]})\frac{c}{\lambda[1-F(a^*(c,T))]}. \tag{12}$$

The limit of (12) as T approaches infinity is 0.

□

A.3 Proofs for the Head Start Case

First, we state two crucial lemmas for the whole section.

Lemma A.5.

1. *For $a_1^I > a^*(c,T)$, not to search is Firm 1's strictly dominant strategy.*
2. *For $a_1^I = a^*(c,T)$, not to search is Firm 1's weakly dominant strategy. If Firm 2 searches with cut-off a_1^I, Firm 1 is indifferent between not to search and search with a_1^I as the cut-off; Otherwise, Firm 1 strictly prefers not to search.*

Proof of Lemma A.5. Suppose Firm 2 does not search, Firm 1's best response is not to search. Suppose Firm 2 searches with cut-off $a_2 \geq a_1^I$. If Firm 1 searches with cut-off $a_1 \geq a_1^I$, following from Lemma A.3, the instantaneous gain from searching for Firm 1 in any state $a_1 \geq a_1^I \geq a^*(c,T)$ is

$$\lambda\int_{a_1}^1 [Z(a|a_2,T)-Z(a_1|a_2,T)]dF(a)-c$$
$$\leq\lambda\int_{a^*(c,T)}^1 [Z(a|a^*(c,T),T)-Z(a^*(c,T)|a^*(c,T),T)]dF(a)-c, \tag{13}$$

where equality holds if and only if $a_1 = a_2 = a^*(c,T)$. The right hand side of inequality (13) is less than or equal to 0 (it equals to 0 iff $c \geq \frac{1}{2}\lambda[1-e^{-\lambda T}]$). Hence, the desired results follow. □

Lemma A.6.

1. *For $a_1^I > F^{-1}(1-\frac{c}{\lambda})$, not to search is Firm 2's strictly dominant strategy.*
2. *For $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, not to search is Firm 2's weakly dominant strategy. If Firm 1 does not search, Firm 2 is indifferent between not to search and search with any $\hat{a}_2 \in [a_2^I, a_1^I]$ as the cut-off. If Firm 1 searches, Firm 2's strictly prefers not to search.*

Proof of Lemma A.6. If Firm 1 does not search, the instantaneous gain from searching for Firm 2 in a state $a_2 \leq a_1^I$ is

$$\lim_{\delta\to 0}\frac{\lambda\delta e^{-\lambda\delta}[1-F(a_1^I)]+o(\delta)-c\delta}{\delta} = \lambda[1-F(a_1^I)]-c\begin{cases}<0 & \text{in Case [1]}\\ =0 & \text{in Case [2].}\end{cases}$$

If Firm 1 searches, Firm 2's instantaneous gain is even lower. Hence, the desired results follow. □

Proof of Theorem 3.1. [1],[2], and [3] directly follow from Lemmas A.5 and A.6. We only need to prove [4] and [5] in the following.

[4]. Following Lemma A.5, if Firm 2 searches with cut-off a_1^I, Firm 1 has two best responses: not to search and search with cut-off a_1^I. If Firm 1 searches with cut-off a_1^I, the instantaneous gain from searching for Firm 2 is

$$\begin{aligned}
&\lambda\int_{a_1^I}^1 Z(a|a_1^I,T)dF(a)-c\\
=&\lambda\int_{a_1^I}^1\left[e^{-\lambda T[1-F(a_1^I)]}+(1-e^{-\lambda T[1-F(a_1^I)]})\frac{F(a)-F(a_1^I)}{1-F(a_1^I)}\right]dF(a)-c\\
=&\frac{1}{2}\lambda(1+e^{-\lambda T[1-F(a_1^I)]})[1-F(a_1^I)]-c\\
>&\lambda[1-F(a_1^I)]-c\\
>&0
\end{aligned}$$

if it is in a state $a_2 < a_1^I = a^*(c,T)$; it is

$$\lambda\int_{a_2}^1[Z(a|a_1^I,T)-Z(a_2|a_1^I,T)]dF(a)-c<0$$

if it is in a state $a_2 > a_1^I = a^*(c,T)$. Hence, the two prescribed strategy profiles are equilibria.

[5]. First, there is no equilibrium in which either firm does not search. If Firm 2 does not search, Firm 1's best response is not to search. However, if Firm 1 does not search, Firm 2's best response is to search with cut-off a_1^I rather than not to search. If Firm 2 searches with cut-off a_1^I, then not to search is not Firm 1's best response, because the instantaneous gain from searching for Firm 1 in state a_1^I is

$$\begin{aligned}&\lambda \int_{a_1^I}^{1} [Z(a|a_1^I,T) - Z(a_1^I|a_1^I,T)]dF(a) - c\\ =&\frac{1}{2}\lambda(1-e^{-\lambda T[1-F(a_1^I)]})[1-F(a_1^I)] - c\\ >&0,\end{aligned}$$

where inequality holds because $a_1^I < a^*(c,T)$.

Next, we argue that there is no equilibrium in which either firm searches with cut-off a_1^I. Suppose Firm i searches with cut-off a_1^I. Firm j's best response is to search with a cut-off $\hat{a}_j \in [a_1^I, a^*(c,T))$. This is because the instantaneous gain from searching for Firm j in a state $a' \geq a_1^I$ is

$$\lambda \int_{a'}^{1} p_2[Z(a|a_1^I,T) - Z(a'|a_1^I,T)]dF(a) - c. \tag{14}$$

(14) is larger than 0 when $a' = a_1^I$. It is less than 0 if $a' = a^*(c,T)$, because by Lemma A.3 we have

$$\begin{aligned}&\lambda \int_{a^*}^{1} p_2[Z(a|a_1^I,T) - Z(a^*|a_1^I,T)]dF(a) - c\\ <&\lambda \int_{a^*(c,T)}^{1} [Z(a|a^*(c,T),T) - Z(a^*|a^*(c,T),T)]dF(a) - c\\ =&0.\end{aligned}$$

Then, the intermediate value theorem and the strict monotonicity yield the unique cut-off value of $\hat{a}_j \in (a_1^I, a^*(c,T))$.

However, if Firm j searches with cut-off $\hat{a}_j \in [a_1^I, a^*(c,T))$, Firm i's best response is to search with a cut-off value $\hat{a}_i \in (\hat{a}_j, a^*(c,T))$ rather than a_1^I, because the instantaneous gain from searching for Firm i in a state $\tilde{a}$ is

$$\lambda \int_{\tilde{a}}^{1} [Z(a|\hat{a}_1,T) - Z(\tilde{a}|\hat{a}_1,T)]dF(a) - c \begin{cases} < 0 & \text{for } \tilde{a} = a^*(c,T) \\ > 0 & \text{for } \tilde{a} = \hat{a}_1 \end{cases}$$

and it is monotone w.r.t. $\tilde{a}$. This results in contradiction. Hence, there is no equilibrium in which either firm searches with a_1^I as the cut-off.

Last, we only need to consider the case in which each firm searches with a cut-off higher than a_1^I. Following the same argument as in the proof of Theorem 2.1, we have $(a^*(c,T), a^*(c,T))$ being the unique equilibrium.

□

Proof of Proposition 4.1. We apply Theorem 3.1 here for the analysis. We only need to show the case for $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$. In this case, Firm 1 does not to search and Firm 2 searches with a_1^I as the cut-off. Now, take the limit $a_1^I \to a^*(c,T)$ from the right hand side of $a^*(c,T)$. In the limit, where Firm 2 searches with $a^*(c,T)$ as the cut-off, Firm 1 weakly prefers not to search. If $a_1^I = a^*(c,T)$, Firm 1 is actually indifferent between searching and not. Hence, a head start in the limit makes Firm 1 weakly better off. Firm 1's payoff when it does not search is $e^{-\lambda T[1-F(a_1^I)]}$, the probability of Firm 2 ending up in a state below a_1^I, is strictly increasing in a_1^I. Hence, a higher value of the head start makes Firm 1 even better off. □

Proof of Proposition 4.2. [1]. For T being small, $a^*(c,T) = 0$. $D^M(0, a_1^I) = 0$, and the partial derivative of $D^M(T, a_1^I)$ w.r.t. T when T is small is

$$\frac{\partial D^M(T, a_1^I)}{\partial T} = \lambda(1-a_1^I)e^{-\lambda T(1-a_1^I)}[1 - \frac{c}{\lambda(1-a_1^I)}] - \lambda e^{-2\lambda T} + ce^{-\lambda T},$$

which equals to $-\lambda a_1^I < 0$ at the limit of $T = 0$.

[2]. Follows from Propositions 3.1 and 4.1. □

Proof of Proposition 4.3. $D^M(T, a_1^I)$ is strictly decreasing in a_1^I, and it goes to the opposite of (12), which is less than 0, as a_1^I goes to $F^{-1}(1-\frac{c}{\lambda})$, and

$$(1 - e^{-\lambda T[1-F(a^*(c,T))]}) - \frac{1}{2}(1 - e^{-2\lambda T}) \tag{15}$$

as a_1^I goes to $a^*(c,T)$. Hence, if (15) is positive, Case 1 yields from the intermediate value theorem; Case 2 holds if (15) is negative. □

A.4 Proofs for the Extended Models

Proof of Proposition 5.1. We argue that, to determine a subgame perfect equilibrium, we only need to consider two kinds of strategies profiles:

a Firm 1 retains its initial innovation and does not search, and Firm 2 searches with a_1^I as the cut-off;

b Firm 1 discards its initial innovation and searches with a_2^I as the cut-off, and Firm 2 retains its initial innovation.

First, suppose $c < \frac{\lambda}{2}(1 + e^{-\lambda T})$. If Firm 1 retains the initial innovation, it will have no incentive to search, and Firm 2 is indifferent between discarding the initial innovation and not. In either case, Firm 2 searches with a_1^I as the cut-off. Given that Firm 1 has discarded its initial innovation, Firm 2 has no incentive to discard its initial innovation as shown in Proposition 4.1.

Second, suppose $c > \frac{\lambda}{2}(1 + e^{-\lambda T})$. In the subgame in which both firms discard their initial innovation, there are two equilibria, in each of which one firm searches with 0 as the cut-off and the other firm does not search. Hence, to determine a subgame perfect equilibrium, we have to consider another two strategy profiles, in addition to [a] and [b]:

c Firm 1 discards its initial innovation and searches with 0 as the cut-off, and Firm 2 discards its initial innovation and does not search.

d Firm 1 discards its initial innovation and does not search, and Firm 2 discards its initial innovation and searches with 0 as the cut-off.

However, we can easily rule out [c] and [d] from the candidates for equilibria. In [c], Firm 2 obtains a payoff of 0. It can deviate by retaining its initial innovation so as to obtain a positive payoff. Similarly, in [d], Firm 1 can deviate by retaining its initial innovation to obtain a positive payoff rather than 0.

Last, it remains to compare Firm 1's payoff in [a] and [b]. In [a], Firm 1's payoff is

$$e^{-\lambda T[1-F(a_1^I)]}. \tag{16}$$

In [b], it is

$$(1 - e^{-\lambda T[1-F(a_2^I)]})(1 - \frac{c}{\lambda[1 - F(a_2^I)]}). \tag{17}$$

The difference between these two payoffs, (17) and (16), is increasing in T, and it equals -1 when $T = 0$ and goes to $1 - \frac{c}{\lambda[1-F(a_2^I)]} > 0$ as T approaches infinity. Hence, the desired result is implied by the intermediate value theorem. □

Proof of Proposition 5.2. The backward induction is similar to the proof of Proposition 5.1, and thus is omitted. □

Proof of Proposition 5.3. The equilibrium for the subgame starting from time t_0 derives from Theorem 3.1. Suppose at time t_0, Firm i is in a state a_i^0, where $\max\{a_1^0, a_2^0\} \geq a_1^I$. Assume $a_i^0 > a_j^0$. If $a_i^0 > F^{-1}(1 - \frac{c}{\lambda})$, then Firm i obtains a continuation payoff of 1, and Firm j obtains 0. If $a_i^0 \in (a^*(c, T-t_0), F^{-1}(1-\frac{c}{\lambda}))$, then Firm i obtains a continuation payoff of $e^{-\lambda(T-t_0)[1-F(a_i^0)]}$, and Firm j obtains $(1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})$.

To prove this result, we first show that not to search before t_0 is Firm 2's best response regardless of Firm 1's action before time t_0. It is equivalent to showing that not to search before t_0 is Firm 2's best response if Firm 2 knows that Firm 1 is definitely going to be in any state $a_1^0 \geq a_L^*$ at time t_0.

As we have shown before, for any $a_1^0 \geq (>) F^{-1}(1-\frac{c}{\lambda})$, Firm 2 (strictly) prefers not to conduct searching before time t_0.

If $a_1^0 \in [a_L^*, F^{-1}(1-\frac{c}{\lambda}))$, Firm 2's unique best response before time t_0 is not to search. The instantaneous gain from searching at any time point before t_0 for Firm 2 in a state below a_1^0 is

$$\begin{aligned}
&\lambda\big[[1-F(F^{-1}(1-\frac{c}{\lambda}))] + \int_{a_1^0}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]} dF(a) \\
&\qquad - [1-F(a_1^0)](1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})\big] - c \\
=&\lambda\big[\int_{a_1^0}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]} dF(a) \\
&\qquad - [1-F(a_1^0)](1-e^{-\lambda(T-t_0)[1-F(a_i^0)]})(1-\frac{c}{\lambda[1-F^{-1}(a_i^0)]})\big],
\end{aligned}$$

which is strictly negative when $T-t_0$ is sufficiently large, and thus conducting a search before time t_0 actually makes Firm 2 strictly worse off in this case.

Next, we show that Firm 1's best response before time t_0 is to search with $F^{-1}(1-\frac{c}{\lambda})$ as the cut-off, if Firm 2 does not search before t_0. To see this, look at the instantaneous gain from searching for Firm 1 in a state below $F^{-1}(1-\frac{c}{\lambda})$:

$$\begin{aligned}
&\lambda\big[[1-F(F^{-1}(1-\frac{c}{\lambda}))] \\
&\qquad + \int_{a_1^I}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]} dF(a) - [1-F(a_1^I)]e^{-\lambda(T-t_0)[1-F(a_1^I)]}\big] - c \\
=&\int_{a_1^I}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]} dF(a) - [1-F(a_1^I)]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
>&\int_{\tilde{a}}^{F^{-1}(1-\frac{c}{\lambda})} e^{-\lambda(T-t_0)[1-F(a)]} dF(a) - [1-F(\tilde{a})]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
>&(1-\frac{c}{\lambda}-F(\tilde{a}))e^{-\lambda(T-t_0)[1-F(\tilde{a})]} - [1-F(\tilde{a})]e^{-\lambda(T-t_0)[1-F(a_1^I)]} \\
=&e^{-\lambda(T-t_0)[1-F(a_1^I)]}[1-\frac{c}{\lambda}-F(\tilde{a})]\left(e^{\lambda(T-t_0)[F(\tilde{a})-F(a_1^I)]} - \frac{1-F(\tilde{a})}{1-\frac{c}{\lambda}-F(\tilde{a})}\right),
\end{aligned}$$

where $\tilde{a}$ is any value in $(a_1^I, F^{-1}(1-\frac{c}{\lambda}))$. The term on the right hand side of the last equality is strictly positive if $T-t_0$ is sufficiently large. Hence, the desired result yields.

□

A.5 Proofs for the Case with Asymmetric Costs

Proposition A.1. *If* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1-e^{-\lambda T})$ *there exists a pure strategy equilibrium* $(a_1^*,\ a_2^*)$ *with* $a_1^*, a_2^* \geq 0$.

Proof of Proposition A.1. We prove the existence of equilibrium by applying Brouwer's fixed point theorem. First, same as in the previous proofs, if Firm j searches with a cut-off $\hat{a}_j \geq 0$, the instantaneous gain from searching for Firm i in state 0 is

$$\lambda \int_0^1 Z(a|1,T) - c_i > 0,$$

and thus Firm i is better off continuing searching if it is in state 0.

Next, let us define for each Firm j a critical value

$$\alpha_j = \sup\{a_j \in [0,1] \mid I(0|\alpha_j, c_i) = \lambda \int_0^1 [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i > 0\}.$$

Suppose there is a $\alpha_j \in (0,1)$ such that

$$I(0|\alpha_j, c_i) = \lambda \int_0^1 [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i = 0.$$

For any $\hat{a}_j \in [0, \alpha_j]$,

$$I(0|\hat{a}_j, c_i) \geq 0 \text{ and}$$
$$I(1|\hat{a}_j, c_i) < 0.$$

By the intermediate value theorem and the strict monotonicity of $Q(a|\hat{a}_j, c_i)$ in a, there must exist a unique $\tilde{a}_i \in [0,1)$ such that

$$I(\tilde{a}_i|\hat{a}_j, c_i) = 0.$$

That is, if Firm j searches with cut-off $\hat{a}_j$, Firm i's best response is to search with cut-off $\tilde{a}_i$.

For any $\hat{a}_j \in (\alpha_j, 1]$, if the set is not empty,

$$I(0|\hat{a}_j, c_i) < 0.$$

That is, Firm i's best response is to search with cut-off 0.

Then, we could define two best response functions $BR_i : [0, 1] \to [0, 1]$ where

$$BR_i(\hat{a}_j) := \begin{cases} 0 & \text{for } \hat{a}_j \in (\alpha_j, 1] \text{ if it is not empty} \\ \tilde{a}_i & \text{where } I(\tilde{a}_i|\hat{a}_j, c_i) = 0 \text{ for } \hat{a}_j \in [0, \alpha_j]. \end{cases}$$

It is also easy to verify that BR_i is a continuous function over $[0, 1]$. Hence, we have a continuous self map $BR : [0, 1]^2 \to [0, 1]^2$ where

$$BR = (BR_1, BR_2)$$

on a compact set, and by Brouwer's fixed point theorem, there must exist of a pure strategy equilibrium in which each Firm searches with a cut-off higher than or equal to 0. □

Proof of Proposition 6.1. First, using the same arguments as in the proof of Proposition 2.1, we claim that if there exists an equilibrium it must be the case that each firm searches with a cut-off higher than or equal to 0 with one strictly positive value for one firm.

Next, we show that there can be no equilibrium in which Firm 2 searches with a cut-off $\hat{a}_2 > 0$ and Firm 1 searches with cut-off 0. Such a strategy profile $(0, \hat{a}_2)$ is an equilibrium if and only if

$$\lambda \int_0^1 [Z(a|\hat{a}_2, T) - Z(0|\hat{a}_2, T)]dF(a) - c_1 \leq 0, \quad \text{and}$$
$$\lambda \int_{\hat{a}_2}^1 [Z(a|0, T) - Z(\hat{a}_2|0, T)]dF(a) - c_2 = 0.$$

However,

$$\begin{aligned} 0 &= \lambda \int_{\hat{a}_2}^1 [Z(a|0, T) - Z(\hat{a}_2|0, T)]dF(a) - c_2 \\ &< \lambda \int_{\hat{a}_2}^1 [Z(a|\hat{a}_2, T) - Z(\hat{a}_2|\hat{a}_2, T)]dF(a) - c_2 \\ &< \lambda \int_0^1 [Z(a|\hat{a}_2, T) - Z(0|\hat{a}_2, T)]dF(a) - c_1 \leq 0, \end{aligned}$$

resulting in a contradiction.

Next, we derive the necessary and sufficient conditions for the existence of an equilirbium in which Firm 2 searches with a cut-off 0 and Firm 1 searches with a cut-off strictly higher than 0. A pair of cut-off rules $(\hat{a}_1, 0)$, $\hat{a}_1 > 0$, is an equilibrium if and only if

$$\lambda \int_{\hat{a}_1}^{1} [Z(a|0,T) - Z(\hat{a}_1|0,T)]dF(a) - c_1 = 0 \quad \text{and} \tag{18}$$

$$\lambda \int_{0}^{1} [Z(a|\hat{a}_1,T) - Z(0|\hat{a}_1,T)]dF(a) - c_2 \leq 0, \tag{19}$$

where

$$(18) \Leftrightarrow \frac{1}{2}\lambda(1-e^{-\lambda T})[1-F(\hat{a}_1)]^2 - c = 0 \Leftrightarrow \hat{a}_j = F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right). \tag{20}$$

Then, (20) and (19) together imply that $(\hat{a}_i, 0)$ is an equilibrium if and only if

$$I\left(0|F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) \leq 0. \tag{21}$$

We will see that if (21) holds there is no other equilibrium.

When (21) does not hold, there is a unique equilibrium, in which each firm searches with a cut-off strictly higher than 0. Because by Proposition A.1 an equilibrium must exists. Let (a_1^*, a_2^*) be such an equilibrium. We first show that $a_1^* > a_2^*$ must hold by proof by contradiction, and then we show that it must be a unique equilibrium. Such a pair (a_1^*, a_2^*) is an equilibrium if and only if

$$\lambda \int_{a_i^*}^{1} [Z(a|a_j^*,T) - Z(a_i^*|a_j^*,T)]dF(a) = c_i \quad \text{for } i = 1,2 \text{ and } j \neq i. \tag{22}$$

Suppose $a_1^* \leq a_2^*$. Applying Lemma A.3, we have

$$\begin{aligned} c_1 =& \lambda \int_{a_1^*}^{1} [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]dF(a) \\ \geq& \lambda \int_{a_2^*}^{1} [Z(a|a_2^*,T) - Z(a_2^*|a_2^*,T)]dF(a) \\ \geq& \lambda \int_{a_2^*}^{1} [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]dF(a) = c_2, \end{aligned}$$

resulting in a contradiction.

Then, we show the uniqueness of the equilibrium for Cases [1] – [3] by contradiction. For Case [1] we show that the solution to (22) is unique, and for Cases [2] and [3] we show that there can be no equilibrium in which each firm searches with a cut-off higher than 0 coexisting with equilibrium $\left(F^{-1}\left(1-\sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$. We can prove all of them together. Suppose there are two equilibria (a_1^*, a_2^*) and $(\tilde{a}_1^*, \tilde{a}_2^*)$, where (a_1^*, a_2^*) is a solution to (22) and $(\tilde{a}_1^*, \tilde{a}_2^*)$ is either $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$ or a solution to (22). It is sufficient to show that the following two cases are not possible:

1. $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$ and
2. $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$.

Suppose $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma A.3 we have

$$\begin{aligned}
0 =& \lambda \int_{a_1^*}^1 [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)]dF(a) - c_1 \\
<& \lambda \int_{\tilde{a}_1^*}^1 [Z(a|a_2^*, T) - Z(\tilde{a}_1^*|a_2^*, T)]dF(a) - c_1 \\
<& \lambda \int_{\tilde{a}_1^*}^1 [Z(a|\tilde{a}_2^*, T) - Z(\tilde{a}_1^*|\tilde{a}_2^*, T)]dF(a) - c_1 = 0,
\end{aligned}$$

resulting in a contradiction.
Suppose $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma A.3 again, we have

$$\begin{aligned}
0 \geq& \lambda \int_{\tilde{a}_2^*}^1 [Z(a|\tilde{a}_1^*, T) - Z(\tilde{a}_2^*|\tilde{a}_1^*, T)]dF(a) - c_2 \\
>& \lambda \int_{a_2^*}^1 [Z(a|\tilde{a}_1^*, T) - Z(a_2^*|\tilde{a}_1^*, T)]dF(a) - c_2 \\
>& \lambda \int_{a_2^*}^1 [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)]dF(a) - c_2 = 0,
\end{aligned}$$

resulting in another contradiction.

□

Proof of Proposition 6.2. For fixed c_2 we have

$$\frac{\partial a_2^*}{\partial a_1^*} = -\frac{\frac{\partial \int_{a_2^*}^1 [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]dF(a)}{\partial a_1^*}}{\frac{\partial \int_{a_2^*}^1 [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]dF(a)}{\partial a_2^*}} = \frac{\int_{a_2^*}^1 \frac{\partial [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]}{\partial a_1^*} dF(a)}{\frac{\partial Z(a_2^*|a_1^*,T)}{\partial a_2^*}} < 0.$$

Then,

$$\begin{aligned}
\frac{\partial a_1^*}{\partial c_1} &= -\frac{-1}{\lambda \int_{a_1^*}^1 \left[\frac{\partial [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]}{\partial a_2^*} \frac{\partial a_2^*}{\partial a_1^*} - \frac{\partial Z(a_1^*|a_2^*,T)}{a_1^*}\right] dF(a)} < 0 \text{ and} \\
\frac{\partial a_2^*}{\partial c_1} &= \frac{\partial a_2^*}{\partial a_1^*} \frac{\partial a_1^*}{\partial c_1} > 0.
\end{aligned}$$

For fixed c_1 we have

$$\frac{\partial a_1^*}{\partial a_2^*} = -\frac{\frac{\partial \int_{a_1^*}^1 [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]dF(a)}{\partial a_2^*}}{\frac{\partial \int_{a_1^*}^1 [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]dF(a)}{\partial a_1^*}} = \frac{\int_{a_1^*}^1 \frac{\partial [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]}{\partial a_2^*} dF(a)}{\frac{\partial Z(a_1^*|a_2^*,T)}{\partial a_1^*}} > 0.$$

Then,

$$\frac{\partial a_2^*}{\partial c_2} = -\frac{-1}{\lambda \int_{a_2^*}^1 \left[\frac{\partial [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]}{\partial a_1^*} \frac{\partial a_1^*}{\partial a_2^*} - \frac{\partial Z(a_2^*|a_1^*,T)}{a_2^*} \right] dF(a)} < 0 \text{ and}$$

$$\frac{\partial a_1^*}{\partial c_1} = \frac{\partial a_1^*}{\partial a_2^*} \frac{\partial a_2^*}{\partial c_2} < 0.$$

□